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Masterarbeit

Del Pezzo Surface Fibrations of Degree 4

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1 Introduction

We study del Pezzo surfaces of degree 4 over fields and del Pezzo surface fibrations of degree 4 over the projective line $\mathbb{P}^1_k$ and the scheme $\operatorname{Spec} \mathbb{Z}$. We start off by summing up some important properties of del Pezzo surfaces of degree 4. We show that roughly all homogeneous polynomials of degree 5 occur (up to scalar multiple) as the characteristic form of some del Pezzo surface of degree 4. In section 2, we describe a few techniques for deciding whether a variety contains a k-rational point. We also give a few examples of del Pezzo surfaces of degree 4 over $\mathbb{Q}$ together with an easy way to show that the Galois group action on the Picard group is maximal. We then give a digest of some results of [HT14], using the techniques we developed in section 2. Moreover, we provide an outlook for del Pezzo surfaces of degree 4 over the integers, including a very short overview of Arakelov theory.

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1.1 General notation

Let k be a field.

As usually, for a vector space V over k, we define the projectivization $\mathbb{P}(V) = (V \setminus 0)/k^\times$ to be the set of lines through the origin. For a vector $v \in V \setminus 0$, let $[v] \in \mathbb{P}(V)$ be the line going through it. We also abbreviate $\mathbb{P}^n_k = \mathbb{P}(k^{n+1})$ and often regard this as a scheme over $\operatorname{Spec} k$. The line going through a point $(x_1, \dots, x_{n+1}) \in k^{n+1} \setminus 0$ is also written as $(x_1 : \dots : x_{n+1})$.

The variety $\operatorname{Gr}(r, V)$ of subspaces of dimension r of the vector space V is called the *dimension r Grassmanian of V*.

We denote by $M_n(k)$ the vector space of $n \times n$-matrices over k. When writing down matrices, we often omit (some) entries that are zero to increase readability. We write zeros, though, when necessary to ensure that columns or rows aren't completely empty. For example,

$$\begin{pmatrix} 0 & 1 & \\ & 2 & 3 \\ & & 4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{pmatrix}.$$

For a matrix $M \in M_n(k)$, we write M^t for its transpose.

1.2 Quadrics

Let us first review the classical theory of quadric surfaces (see for example [Sch13] and [Rei72]).

We denote by $\mathrm{Sym}_n(k) \subseteq M_n(k)$ the subspace of symmetric $n \times n$-matrices.

Let now k be a field of characteristic $\mathrm{char}(k) \neq 2$.

To a symmetric matrix $Q \in \mathrm{Sym}_n(k)$, we can associate the symmetric bilinear form $\mathfrak{q} : k^n \times k^n \to k$ given by $\mathfrak{q}(x, y) = x^t Q y = y^t Q x$ and the quadratic form $\mathfrak{q} : k^n \to k$ given by $\mathfrak{q}(x) = \mathfrak{q}(x, x) = x^t Q x$. The quadratic form $\mathfrak{q}(x)$ is a homogeneous polynomial of degree 2 in the coordinates $x_1, \dots, x_n$ of $x \in k^n$. We will also write Q for the zero set $V(\mathfrak{q}) \subset \mathbb{P}_k^{n-1}$ of the quadratic form $\mathfrak{q}$. This way, we obtain a map from the set $\mathbb{P}(\mathrm{Sym}_n(k))$ to the set of hypersurfaces in $\mathbb{P}_k^{n-1}$. Hypersurfaces arising this way are called *quadrics* or *quadric surfaces*. When changing the basis of $\mathbb{P}_k^{n-1}$ using a matrix T, the matrix Q has to be replaced by $T^t Q T$.

Lemma 1.2.1. *A quadric $Q \subseteq \mathbb{P}_k^{n-1}$ is smooth if and only if $\ker(Q) \neq 0$.*

Proof. The derivative vector of $\mathfrak{q}$ at a point $x \in \overline{k}^n \setminus 0$ is $D\mathfrak{q}(x) = Qx$. Thus, $[x]$ is a singular point if and only if $Qx = 0$ (it must then also lie on the variety Q since $\mathfrak{q}(x) = x^t Q x = 0$). □

For a smooth quadric $Q \subseteq \mathbb{P}_k^{n-1}$, we define the *discriminant class* $d(Q) = [\det(Q)] \in k^\times / k^{\times 2}$ if n is even. It is then invariant under base changes on $\mathbb{P}_k^{n-1}$ and doesn't change when multiplying the matrix Q by a constant $\lambda \in k^\times$.

For any quadric $Q \subseteq \mathbb{P}_k^{n-1}$ with $d = \dim(\ker(Q))$, there exists a plane $H \cong \mathbb{P}_k^{n-1-d}$ in $\mathbb{P}_k^{n-1}$ such that $Q \cap H$ is a smooth quadric on $H \cong \mathbb{P}_k^{n-1-d}$. We then define the *smooth discriminant class* $\varepsilon(Q) = d(Q \cap H) \in k^\times / k^{\times 2}$ if $n - d$ is even. It is invariant under base changes on $\mathbb{P}_k^{n-1}$, and independent of the choice of H and of isomorphisms $H \cong \mathbb{P}_k^{n-1-d}$. It doesn't change when multiplying the matrix Q by a constant $\lambda \in k^\times$.

Finally, let us recall some simple geometric properties of quadrics in low dimension (see [Rei72, Theorems 1.2 and 1.8] and [Sch13, Theorem 1.5]).

Lemma 1.2.2. *Let $Q \subseteq \mathbb{P}_k^3$ be a smooth quadric. Then, there are two families of lines defined over $k(\sqrt{\varepsilon(Q)})$ such that through any point $x \in Q$ passes exactly one line of each family. Each line on Q lies in one of the families.*

Lemma 1.2.3. *Let $Q \subseteq \mathbb{P}_k^4$ be a quadric of rank 4 (so the kernel of Q has dimension 1). Then, there are two families of planes defined over $k(\sqrt{\varepsilon(Q)})$ such that through any nonsingular point $x \in Q$ passes exactly one plane of each family. Each plane on Q contains the singular point on Q and lies in one of the families.*

1.3 Intersections of two quadrics

See [Rei72] or [Sch13] for an introduction to the theory of intersections of two quadrics. We will repeat some definitions and basic properties.

Let Q be a line in the projective space $\mathbb{P}(\mathrm{Sym}_n(k))$ (a *pencil of quadrics*)[1]. Choose two points $[Q_\infty], [Q_0] \in Q$ on this line. We then call the intersection $Q_\infty \cap Q_0 \subset \mathbb{P}_k^{n-1}$ of quadrics the *variety associated to* Q. All points on the line Q are of the form $[Q_r]$ with $Q_r = uQ_\infty + vQ_0$ for some $r = (u : v) \in \mathbb{P}_k^1$. But $Q_\infty \cap Q_0 \subset Q_r$, so the intersection doesn't depend on the choice of Q_∞ and Q_0. An important invariant of a pencil Q of quadrics (with chosen Q_∞ and Q_0) is the *characteristic form* $\chi(Q)$, which is the homogeneous polynomial of degree n in the variables u and v given by $\chi(Q)(u,v) = \det(Q_{(u:v)}) = \det(uQ_\infty + vQ_0)$, and in particular its set of zeros in $\mathbb{P}_{\overline{k}}^1$. Choosing different matrices Q_∞ and Q_0 on the pencil Q corresponds to a linear automorphism of $\mathbb{P}_k^1$.

If the matrix Q_∞ is invertible, then the characteristic form $\chi(Q)(u,1)$ is the characteristic polynomial of the matrix $-Q_0Q_\infty^{-1}$. Many well-known statements about characteristic polynomials carry over to the characteristic form:

Lemma 1.3.1. *a) The point* $r \in \mathbb{P}^1$ *is a zero of* $\chi(Q)$ *if and only if* $\ker(Q_r) \neq 0$*, i.e., if and only if the quadric* Q_r *is not smooth.*

b) If $\chi(Q) \neq 0$*, then* $\ker(Q_\infty) \cap \ker(Q_0) = 0$.

c) Assume $\chi(Q) \neq 0$*. Let* $r_1, \dots, r_m \in \mathbb{P}_{\overline{k}}^1$ *be the zeros of* $\chi(Q)$ *over the algebraic closure* $\overline{k}$ *and let* $a_1, \dots, a_m$ *be the corresponding multiplicities. The kernels form a direct sum* $\ker(Q_{r_1}) \oplus \cdots \oplus \ker(Q_{r_m})$*. Furthermore,* $1 \leq \dim\ker(Q_{r_i}) \leq a_i$ *for all* i.

Proof. a) This is a well-known fact about determinants.

b) This immediately follows from the fact that $\ker(Q_\infty) \cap \ker(Q_0) \subseteq \ker(uQ_\infty + vQ_0) = 0$ for some non-root $(u : v) \in \mathbb{P}_{\overline{k}}^1$ of $\chi(Q)$.

c) Since $\chi(Q) \neq 0$, there exists some $r \in \mathbb{P}_{\overline{k}}^1$ such that $\ker(Q_r) = 0$. By using a different choice of Q_∞ and Q_0, we can therefore assume that Q_∞ is invertible. This transforms the statement into well-known facts about the characteristic polynomial of $-Q_0Q_\infty^{-1}$. □

Definition 1.3.2. We call the variety associated to a pencil Q of quadrics in the four-dimensional projective space $\mathbb{P}_k^4$ a *del Pezzo surface of degree 4* if it is smooth.

Recall that k^n possesses an orthogonal basis with respect to any symmetric bilinear form $\mathfrak{q}$, i.e. any quadric $Q \in \mathrm{Sym}_n(k)$ can be "diagonalized": there exists some $T \in \mathrm{GL}_n(k)$ such that all nonzero entries of T^tQT lie on the diagonal.

Lemma 1.3.3 (cf. [Sch13, Theorem 1.6])**.** *The variety* $X \subseteq \mathbb{P}_k^n$ *associated to a pencil of quadrics* Q *is smooth if and only if* $\chi(Q)$ *has exactly* $n+1$ *distinct zeros over the algebraic closure* $\overline{k}$ *of* k.

[1] In other words, $Q \in \mathrm{Gr}(2, \mathrm{Sym}_n(k))$, where $\mathrm{Gr}(2, \mathrm{Sym}_n(k))$ is a space of dimension $2(\dim(\mathrm{Sym}_n(k)) - 2) = n(n+1) - 4$.

Proof. We may assume that k is algebraically closed.

The variety X has a singularity at $x \in X(\overline{k})$ if and only if the derivative vectors $D\mathfrak{q}_\infty(x) = 2Q_\infty x$ and $D\mathfrak{q}_0(x) = 2Q_0 x$ are not linearly independent, i.e. there exists some $r \in \mathbb{P}^1_{\overline{k}}$ such that $Q_r x = 0$. This is obviously only possible if r is a zero of $\chi(Q)$. Hence, we are to study when there exists some $x \in X(\overline{k})$ such that $Q_r x = 0$.

Assume without limit of generality that $r = (0:1)$, so $Q_r = Q_0$. We are then interested in when there exists some $x \in \ker(Q_0) \setminus 0$ such that $\mathfrak{q}_\infty(x) = x^t Q_\infty x = 0$ (the first condition already implies that $\mathfrak{q}_0(x) = 0$). By base changing, we can also assume that

$$Q_0 = \begin{pmatrix} a_0 & & & & \\ & a_1 & & & \\ & & \ddots & & \\ & & & a_{n-1} & \\ & & & & a_n \end{pmatrix}$$

is a diagonal matrix. Since $\det(Q_0) = 0$, we may further assume that $a_0 = 0$. Let $Q_\infty = (b_{ij})_{0 \le i,j \le n}$. Now, the linear term of the polynomial $\chi(Q)(u:1) = \det(uQ_\infty + Q_0)$ is $b_{00}a_1 \ldots a_n u$, so $\chi(Q)$ has a simple zero at $r = (0:1)$ if and only if $b_{00}a_1 \ldots a_n \neq 0$.

In that case we have $a_1, \ldots, a_n \neq 0$, so $\ker(Q_0)$ is spanned by the first coordinate vector e_0. But also $\mathfrak{q}_\infty(e_0) = b_{00} \neq 0$, so there is no $x \in \ker(Q_0) \setminus 0$ such that $\mathfrak{q}_\infty(x) = x^t Q_\infty x = 0$. In particular, if all zeros of $\chi(Q)$ are simple, the variety X must be smooth.

On the other hand, if $r = (0:1)$ is a multiple root, then $b_{00}a_1 \ldots a_n = 0$. If $b_{00} = 0$, then $e_0 \in \ker(Q_0)$ and $\mathfrak{q}_\infty(e_0) = b_{00} = 0$, so e_0 is a singularity of X. If $a_i = 0$ for any $i \geq 1$, then $\ker(Q_0)$ contains at least a line in $\mathbb{P}^n_k$. Therefore, since k is algebraically closed, the intersection of $\ker(Q_0)$ and Q_∞ in $\mathbb{P}^n_k$ must be nonempty. But any point in the intersection is a singularity of X. □

Example 1.3.4. The surface $X \subseteq \mathbb{P}^4_{\mathbb{Q}}$ defined by the equations

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$$
$$x_1^2 + 2x_2^2 + x_3^2 + 2x_4^2 + x_3x_4 = 0$$

is a del Pezzo surface of degree 4 over $\mathbb{Q}$. Indeed, it is given by the matrices

$$Q_\infty = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \quad \text{and} \quad Q_0 = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 2 & & \\ & & & 1 & \frac{1}{2} \\ & & & \frac{1}{2} & 2 \end{pmatrix}$$

which yield the characteristic form

$$\chi(Q)(u,v) = \det \begin{pmatrix} u & & & & \\ & u+v & & & \\ & & u+2v & & \\ & & & u+v & \frac{1}{2}v \\ & & & \frac{1}{2}v & u+2v \end{pmatrix} \tag{1.1}$$

$$= u(u+v)(u+2v)\left(u^2 + 3uv + \frac{7}{4}v^2\right), \tag{1.2}$$

that has 5 distinct zeroes $(0:1), (-1:1), (-2:1), \left(\frac{-3+\sqrt{2}}{2} : 1\right), \left(\frac{-3-\sqrt{2}}{2} : 1\right)$.

Lemma 1.3.5 (cf. [Wit07, Theorem 3.28]). *Assume that $\chi(Q)$ has exactly n distinct zeroes $r_1, \ldots, r_n \in \mathbb{P}^1_k$. Then, there is a matrix $T \in \mathrm{GL}_n(k)$ such that $T^t Q_\infty T$ and $T^t Q_0 T$ are both diagonal matrices.*

Proof. According to Lemma 1.3.1, there have to be n linearly independent vectors $t_i \in \ker(Q_{r_i})$. For $i \neq j$, we now have $Q_{r_i} t_i = 0$ and $Q_{r_j} t_j = 0$, so $t_j^t Q_{r_i} t_i = 0$ and $t_j^t Q_{r_j} t_i = t_i^t Q_{r_j} t_j = 0$. As $r_i \neq r_j$, we obtain $t_j^t Q_\infty t_i = 0$ and $t_j^t Q_0 t_i = 0$ as linear combinations of the previous two equations. Therefore, the matrix T with columns t_i satisfies that $T^t Q_\infty T$ and $T^t Q_0 T$ are diagonal matrices. □

Remark 1.3.6. Várilly-Alvarado and Viray [VAV12, Proposition 4.2] showed that we more generally get $T^t Q_\infty T$ and $T^t Q_0 T$ to be block diagonal matrices where the block sizes equal the degrees of the irreducible factors of $\chi(Q)$.

If X is smooth, then the matrix Q_r associated to any root r of $\chi(Q)$ must be of rank n. Thus, if n is even, we can also compute the smooth discriminant class $\varepsilon(Q_r) \in k(r)^\times / k(r)^{\times 2}$ corresponding to this root and it doesn't change when multiplying any matrix by a constant $\lambda \in k^\times$. This class is of importance for instance for computing the Brauer group of a del Pezzo surface X (see section 2.5).

Lemma 1.3.7. *Let X be defined by the diagonal matrices*

$$Q_\infty = \begin{pmatrix} a_0 & & & & \\ & a_1 & & & \\ & & \ddots & & \\ & & & a_{n-1} & \\ & & & & a_n \end{pmatrix} \quad \text{and} \quad Q_0 = \begin{pmatrix} b_0 & & & & \\ & b_1 & & & \\ & & \ddots & & \\ & & & b_{n-1} & \\ & & & & b_n \end{pmatrix}.$$

The characteristic form $\chi(Q)$ is $\prod_i (ua_i + vb_i)$, so its roots are $(-b_0 : a_0), \ldots, (-b_n : a_n) \in \mathbb{P}^1_k$.
If X is smooth and n is even, then the smooth discriminant class corresponding to the root $(-b_i : a_i) \in \mathbb{P}^1_k$ is the class of $\prod_{j \neq i}(a_i b_j - a_j b_i)$ in $k^\times / k^{\times 2}$.

Proof. The formula for the characteristic form immediately follows from the definitions.

To compute the smooth discriminant class corresponding to such a root $r_i = (-b_i : a_i)$ of $\chi(Q)$, we have to choose a basis for a hyperplane H_i not containing the kernel of Q_{r_i}. But since the kernel is spanned by the i-th standard coordinate vector, we can simply take the other n standard coordinate vectors as a basis. The determinant of the resulting matrix $Q_{r_i}|_{H_i}$ is $\prod_{j \neq i}(a_i b_j - a_j b_i)$. □

From Lemmas 1.3.5 and 1.3.7 follows that any two smooth varieties corresponding to pencils of quadrics over an algebraically closed field are isomorphic if they have the same characteristic form. However, this is not true over general fields.

1.4 Notation for global fields

Let k be a global field (i.e., a number field or a finite extension of $\mathbb{F}_q(T)$). Let M_k be the set of places (valuations) of k. For any valuation $v \in M$, let k_v be the completion of k with respect to v, let $\mathcal{O}_v = \{x \in k_v \mid v(x) \geq 0\}$ be the corresponding valuation ring and let $\mathfrak{m}_v \subsetneq \mathcal{O}_v$ be the corresponding maximal ideal. We write $\mathbb{A}_k = \{(x_v) \in \prod_v k_v \mid x_v \in \mathcal{O}_v \text{ for almost all } v\}$ for the ring of adeles.

If k is a number field, we let $\mathcal{O}_k$ be the ring of integers of k.

1.5 Occuring characteristic forms

We now want to study which homogeneous polynomials χ of degree n are the characteristic form $\chi(Q)$ for some pencil Q of quadrics.

Over algebraically closed fields, the answer is obviously "all" according to Lemma 1.3.7.

Over non-closed fields, we will just show that every polynomial χ without roots at $(0 : 1)$ or $(1 : 0)$ is up to a scalar multiple the characteristic polynomial of some pencil Q of quadrics. We will give an explicit way to compute such a pencil of quadrics.

In general, if we set $Q_\infty = I$, we want to find a symmetric matrix Q_0 with given characteristic polynomial $p(x)$. The companion matrix of $p(x)$ is a classical example of a matrix with characteristic polynomial $p(x)$ but it is (almost) never symmetric. The problem for symmetric matrices is significantly harder and has been answered for number fields in case $p(x)$ has an odd degree factor by Bender [Ben68]. Of course we cannot expect to obtain such a symmetrix matrix Q_0 for every polynomial $p(x)$: over a field $k \subseteq \mathbb{R}$, the roots of the characteristic polynomial of a symmetric matrix are all real.

Theorem 1.5.1. *Let $\chi = \sum_{i=0}^{n} p_i u^i v^{n-i}$ be a homogeneous polynomial of degree n in the variables u and v over the field k. Assume $p_n = (-1)^{\lfloor n/2 \rfloor}$ and $p_0 \neq 0$. Then, there exists some pencil Q of quadrics over k such that $\chi = \chi(Q)$.*

Proof. Consider the companion matrix

$$A = \begin{pmatrix} & -1 & & \\ & & \ddots & \\ & & & -1 \\ p_0 & p_1 & \cdots & p_{n-1} \end{pmatrix}$$

of the monic polynomial $(-1)^{\lfloor n/2 \rfloor} \sum_{i=0}^{n} p_i x^i$ (see for example [Neu92, page 10]). We know that the characteristic polynomial $\det(xI - A)$ of A is $(-1)^{\lfloor n/2 \rfloor} \sum_{i=0}^{n} p_i x^i$. Let furthermore

$$B = (b_{ij})_{1 \le i,j \le n} = \begin{pmatrix} & & & & 1 \\ & & & 1 & \frac{p_1}{p_0} \\ & & \iddots & \iddots & \vdots \\ & 1 & \iddots & \iddots & \frac{p_{n-2}}{p_0} \\ 1 & \frac{p_1}{p_0} & \cdots & \frac{p_{n-2}}{p_0} & \frac{p_{n-1}}{p_0} \end{pmatrix}$$

with

$$b_{ij} = \begin{cases} 0, & i + j \le n \\ \frac{p_{i+j-n-1}}{p_0}, & i + j \ge n + 1. \end{cases}$$

Then,

$$BA = (c_{ij})_{1 \le i,j \le n} = \begin{pmatrix} p_0 & p_1 & \cdots & p_{n-2} & p_{n-1} \\ p_1 & \frac{p_1^2}{p_0} & \cdots & \frac{p_1 p_{n-2}}{p_0} & -1 + \frac{p_1 p_{n-1}}{p_0} \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ p_{n-2} & \frac{p_1 p_{n-2}}{p_0} & \cdots & -\frac{p_{n-4}}{p_0} + \frac{p_{n-2}^2}{p_0} & -\frac{p_{n-3}}{p_0} + \frac{p_{n-2} p_{n-1}}{p_0} \\ p_{n-1} & -1 + \frac{p_1 p_{n-1}}{p_0} & \cdots & -\frac{p_{n-3}}{p_0} + \frac{p_{n-2} p_{n-1}}{p_0} & -\frac{p_{n-2}}{p_0} + \frac{p_{n-1}^2}{p_0} \end{pmatrix}$$

with

$$c_{ij} = \begin{cases} \frac{p_{i-1} p_{j-1}}{p_0}, & i + j \le n + 1 \\ \frac{p_{i-1} p_{j-1}}{p_0} - \frac{p_{i+j-n-2}}{p_0}, & i + j \ge n + 2. \end{cases}$$

We have $\det(B) = (-1)^{\lfloor n/2 \rfloor}$.

Now both B and BA are symmetric matrices, so we can look at the pencil of quadrics given by $Q_\infty = B$ and $Q_0 = BA$. Its characteristic form is

$$\chi(Q) = \det(uQ_\infty + vQ_0) = \det(B)\det(uI - vA) = \sum_{i=0}^{n} p_i u^i v^{n-i} = \chi.$$

□

Remark 1.5.2. There is a whole space of symmetric matrices B we can use in the proof of the theorem (for which BA is also symmetric). We can freely choose the first row of B and the remaining entries of the matrix can then be computed using the system of linear equations describing $(BA)^t = BA$.

2 Techniques

Let us first recall some general strategies to decide whether a variety over a field k has k-rational points ($X(k) \neq \emptyset$).

2.1 Tsen ranks

Tsen ranks (see for example [Lor08, Chapter 27]) provide a very easy way to show that a system of polynomial equations has a common nontrivial zero just by counting variables and looking at the degrees of the polynomials. In particular, it doesn't take into account any arithmetic properties of the equations.

Definition 2.1.1. Let k be a field. The *Tsen rank* T_k is the smallest integer $r \geq 0$ such that a system of m polynomial equations of degrees $d_1, \ldots, d_m$ in n variables having the trivial solution $(0, \ldots, 0)$ also has a non-trivial solution whenever $n > d_1^r + \cdots + d_m^r$. If there is no such r, we define $T_k = \infty$.

Remark 2.1.2. The condition holds for all r if we assume $d_1 = \cdots = d_m = 1$: a system of m linear equations in $n > m$ variables has either no or multiple solutions.

Since homogeneous polynomials have the trivial solution $(0, \ldots, 0)$, we obtain the following inportant remark.

Remark 2.1.3. Let k be a field of Tsen rank $T_k < \infty$. Any m homogeneous polynomials of degrees $d_1, \ldots, d_m$ in $n+1$ variables have a common root $x \in \mathbb{P}_k^n$ if $n \geq d_1^r + \cdots + d_m^r$.

Example 2.1.4. Subfields k of $\mathbb{R}$ have Tsen rank ∞ since $x_1^2 + \cdots + x_n^2 = 0$ has no non-trivial solutions for any $n \geq 1$.

Theorem 2.1.5. *A field k has Tsen rank 0 if and only if it is algebraically closed.*

Proof. Assume k is not algebraically closed, i.e., there exists some polynomial $a_d X^d + \cdots + a_0 \in k[X]$ (with $d > 1$ and $a_d \neq 0$) that doesn't have a root in k. Consider the homogeneous polynomial $a_d X_1^d + a_{d-1} X_1^{d-1} X_2 + \cdots + a_0 X_2^d \in k[X_1, X_2]$ of degree d in 2 variables. It obviously has the trivial root $(0, 0)$ but no non-trivial root. Hence, the Tsen rank of k is at least 1 (since $2 > d^0$).

Conversely, assume that k is algebraically closed. According to [Mil12, Theorem 9.2] or [Lor08, Chapter 27, Theorem 1], the dimension of an affine variety decreases by at most one when we intersect with a hypersurface if the intersection is nonempty. The proof of this uses Noether normalization to reduce to the case of affine hypersurfaces. □

Corollary 2.1.6. *Let k be algebraically closed. The intersection of m hypersurfaces in $\mathbb{P}^n_k$ is nonempty if $n \geq m$.*

Theorem 2.1.7 (Chevalley-Warning)**.** *Every finite field $\mathbb{F}_q$ has Tsen rank 1. More precisely: let $p_1, \dots, p_m \in \mathbb{F}_q[X_1, \dots, X_n]$ be polynomials with $d_i := \deg(p_i)$ and assume that $n > d_1 + \cdots + d_m$. Then, the number of common zeros in $\mathbb{F}_q^n$ of the polynomials is divisible by* $\operatorname{char}(\mathbb{F}_q)$.

Proof. For any $0 < e < q-1$, there exists some $y \in \mathbb{F}_q^\times$ such that $y^e \neq 1$ (since $\mathbb{F}_q^\times \cong \mathbb{Z}/(q-1)\mathbb{Z}$). Hence, $\sum_{x \in \mathbb{F}_q} x^e = \sum_{x \in \mathbb{F}_q} (xy)^e = y^e \sum_{x \in \mathbb{F}_q} x^e$ implies that $\sum_{x \in \mathbb{F}_q} x^e = 0$. This equation also holds for $e = 0$ since $q = 0$ in $\mathbb{F}_q$. Therefore, any polynomial $p \in \mathbb{F}_q[X_1, \dots, X_n]$ of (total) degree less than $(q-1)n$ satisfies $\sum_{(x_1,\dots,x_n) \in \mathbb{F}_q^n} p(x_1, \dots, x_n) = 0$ (since the degree of every monomial must be less than $q-1$ with respect to at least one X_i).

The term

$$\prod_{i=1}^{m} \left(1 - p_i(x_1, \dots, x_n)^{q-1}\right)$$

is 1 if $(x_1, \dots, x_n)$ is a common zero of the polynomials $p_1, \dots, p_m$ and 0 otherwise. As a polynomial in $x_1, \dots, x_n$ it has (total) degree at most $d_1(q-1) + \cdots + d_m(q-1) < n(q-1)$. Hence,

$$\sum_{(x_1,\dots,x_n) \in \mathbb{F}_q^n} \prod_{i=1}^{m} \left(1 - p_i(x_1, \dots, x_n)^{q-1}\right) = 0$$

and this must be the image of the number of common zeros in $\mathbb{F}_q^n$ of the polynomials $p_1, \dots, p_m$ under the ring homomorphism $\mathbb{Z} \to \mathbb{F}_q$, so this number must be divisible by $\operatorname{char}(\mathbb{F}_q)$. In particular, if the polynomials have the trivial zero $(0, \dots, 0)$, they must have at least one other zero. □

Theorem 2.1.8. *For every field k of finite Tsen rank T_k, the Tsen rank of $k(T)$ is at most $T_k + 1$.*

Proof. Let $p_1, \dots, p_m \in k(T)[X_1, \dots, X_n]$ be polynomials with $d_i := \deg(p_i)$ having the common zero $(0, \dots, 0)$ such that $n > d_1^{T_k+1} + \cdots + d_m^{T_k+1}$. We now want to construct a solution to the polynomial equations $p_i(x_1, \dots, x_n) = 0$. Let us try $x_i := \sum_{j=0}^{l} y_{ij} T^j$ for $y_{ij} \in k$ and some sufficiently large l. The equation $p_i(x_1, \dots, x_n) = 0$ can then be reinterpreted in terms of the $(l+1)n$ variables y_{ij} and comparing T-coefficients yields at most $ld_i + C$ equations of degree d_i (for some C not depending on l) with coefficients in k. We want to show that, for sufficiently large l, this system of equations has a non-trivial solution. Due to the definition of the Tsen rank of k, it is sufficient to ensure that

$$(l+1)n > \sum_{i=1}^{m} (ld_i + C) d_i^{T_k}.$$

But this inequality holds for sufficiently large l, since $n > d_1^{T_k+1} + \cdots + d_m^{T_k+1}$. □

Lemma 2.1.9. *Let $\mathbb{F}_q$ be a finite field. Then, the Tsen rank of $\mathbb{F}_q(T)$ is 2.*

Proof. By Theorems 2.1.7 and 2.1.8, the Tsen rank of $\mathbb{F}_q(T)$ is at most 2.
Since $\mathbb{F}_q$ has an extension of degree 2, there is some polynomial $X^2 + \lambda X + \mu \in \mathbb{F}_q[X]$ that doesn't have a root (in $\mathbb{F}_q$). Consider the homogeneous equation $TX_1^2 = X_2^2 + \lambda X_2 X_3 + \mu X_3^2$. Assume it has a non-trivial solution $(x_1, x_2, x_3) \in \mathbb{F}_q[T]^3$. Without limit of generality, assume that $\gcd(x_1, x_2, x_3) = 1$. From the equation $Tx_1^2 = x_2^2 + \lambda x_2 x_3 + \mu x_3^2$, we obtain $0 = x_2(0)^2 + \lambda x_2(0) x_3(0) + \mu x_3(0)^2$. Since $X^2 + \lambda X + \mu$ doesn't have a root in $\mathbb{F}_q$, this implies $x_2(0) = x_3(0) = 0$. Hence, both polynomials x_2 and x_3 are divisible by T. That means that $Tx_1^2 = x_2^2 + \lambda x_2 x_3 + \mu x_3^2$ is divisible by T^2, so x_1 is divisible by T. But then, T is a common divisor of $x_1, x_2, x_3 \in \mathbb{F}_q[T]$, contradicting our assumption. Therefore, the homogeneous equation $TX_1^2 = X_2^2 + \lambda X_2 X_3 + \mu X_3^2$ of degree 2 in 3 variables doesn't have a non-trivial zero, so the Tsen rank of $\mathbb{F}_q(T)$ is at least 2. □

Let us now apply the theory of Tsen ranks to intersections $X \subseteq \mathbb{P}_k^4$ of two quadrics (for example del Pezzo surfaces of degree 4). Since there are 5 variables and two homogeneous equations of degree 2, we immediately see that $X(k) \neq \emptyset$ if the Tsen rank of k is at most 1 (for example if k is algebraically closed, the function field over an algebraically closed field or a finite field). But the Tsen rank of a function field over a finite field is 2 and the Tsen rank of $\mathbb{Q}$ is ∞, so in these two cases computing Tsen ranks is not sufficient to show that $X(k) \neq \emptyset$.

2.2 The Arithmetic Zeta Function

Let C be a smooth projective curve over a finite field $k = \mathbb{F}_q$.
Let us start with some well-known generalities about divisors of C.

Lemma 2.2.1 ([Ros02, Lemma 5.5])**.** *For each $n \in \mathbb{Z}$, there are only finitely many divisors $D \geq 0$ of degree $\deg(D) = n$.*

Proof. It is obviously sufficient to show that there are only finitely many *prime* divisors P of degree $\deg(P) = n$ for any n. It is a well-known fact that the finite field k has exactly one extension K of degree n. Since P is a prime divisor of degree n, it must split into n (conjugate) prime divisors of degree 1 over K. But there are obviously just finitely many prime divisors of degree 1 (i.e., points) over K since C can be embedded into a projective space and K is finite. □

Theorem 2.2.2 ([Ros02, Lemma 5.6])**.** *The curve C has only finitely many divisor classes of degree 0 (the group $\mathrm{Pic}^0(C)$ is finite).*

Proof. Fix a divisor $B \in \mathrm{Div}(C)$ satisfying $\deg(B) > 2g - 2$ (e.g. a sufficiently large multiple of a prime divisor). For any $A \in \mathrm{Div}^0(C)$, we thus have $\deg(A + B) > 2g - 2$, so, according to the Riemann-Roch Theorem, we have $l(A + B) > 0$. Hence, for every divisor class d of degree 0, the class $d + B$ contains an effective divisor. This means that the number of divisor classes of degree 0 is at most the number of divisors $D \geq 0$ of degree $\deg(D) = \deg(B)$, which is finite according to the previous lemma. □

Definition 2.2.3. For a divisor D of the curve C, we call $N_C(D) := q^{\deg(D)}$ the *norm of D*.

Definition 2.2.4. Define the *arithmetic zeta function* ζ_C associated to C by

$$\zeta_C(z) = \sum_{0 \leq D \in \mathrm{Div}(C)} N_C(D)^{-z}$$

wherever this converges.

We will now show that ζ_C has a meromorphic continuation and compute its poles.

Theorem 2.2.5 ([Sch31, §8.2a)]). *The sum converges for all $z \in \mathbb{C}$ satisfying $\Re(z) > 1$ and we then have*

$$\zeta_C(z) = \sum_{0 \leq D \in \mathrm{Div}(C)} N_C(D)^{-z} = \prod_{P \in \mathrm{Div}(C)\ prime} \frac{1}{1 - N_C(P)^{-z}}.$$

Furthermore, the function ζ_C defined this way for $\Re(z) > 1$ has a unique continuation to a meromorphic function $\zeta_C : \mathbb{C} \to \mathbb{C}$ that has poles of order 1 at $z = 0$ and $z = 1$ and is holomorphic everywhere else.

Proof. Let h be the number of divisor classes of degree 0. Let $d > 0$ be such that $d\mathbb{Z}$ is the image of the function $\deg : \mathrm{Div}(C) \to \mathbb{Z}$, i.e., d is the greatest common divisor of the degrees of all divisors. For each $k \geq 0$, we can thus choose representatives $C_k^1, \dots, C_k^h$ of degree kd of the divisor classes. Then, the number of divisors $\mathfrak{a} \in |C_k^i|$ is

$$\frac{q^{l(C_k^i)} - 1}{q - 1}.$$

The Riemann-Roch Theorem tells us that

$$l(C_k^i) = \deg(C_k^i) - g + 1 = kd - g + 1$$

for sufficiently large kd (say $k \geq k_0$). Hence,

$$\begin{aligned}
\zeta_C(z) &= \sum_{D \in \mathrm{Div}(C)} N_C(D)^{-z} \\
&= \sum_{k=0}^{\infty} \#\{D \in \mathrm{Div}(C) \mid \deg(D) = kd\} \cdot q^{-kdz} \\
&= \sum_{k=0}^{\infty} \sum_{i=1}^{h} \frac{q^{l(C_k^i)} - 1}{q-1} \cdot q^{-kdz} \\
&= \sum_{k=0}^{k_0-1} \sum_{i=1}^{h} \frac{q^{l(C_k^i)} - 1}{q-1} \cdot q^{-kdz} + \sum_{k=k_0}^{\infty} \sum_{i=1}^{h} \frac{q^{kd-g+1} - 1}{q-1} \cdot q^{-kdz} \\
&= \sum_{k=0}^{k_0-1} \sum_{i=1}^{h} \frac{q^{l(C_k^i)} - 1}{q-1} \cdot q^{-kdz} + \frac{h}{q-1} \sum_{k=k_0}^{\infty} \left(q^{kd(1-z)-g+1} - q^{-kdz}\right) \\
&= \sum_{k=0}^{k_0-1} \sum_{i=1}^{h} \frac{q^{l(C_k^i)} - 1}{q-1} \cdot q^{-kdz} + \frac{h}{q-1} \left(\frac{q^{k_0 d(1-z)-g+1}}{1-q^{d(1-z)}} - \frac{q^{-k_0 dz}}{1-q^{-dz}}\right)
\end{aligned}$$

where the sum $\sum_{k=k_0}^{\infty} \left(q^{kd(1-z)-g+1} - q^{-kdz}\right)$ and thus all sums in the lines before converge absolutely whenever $\Re(z) > 1$. The last term is obviously meromorphic and holomorphic everywhere except at $z = 0$ and $z = 1$. It has a pole of order 1 at $z = 0$ since $1 - q^{-dz} = 0$ but $\frac{\mathrm{d}}{\mathrm{d}z}(1 - q^{-dz}) = 1 + q^{-dz}\log(q)d \neq 0$ at the point $z = 0$ and a pole of order 1 at $z = 1$ since $1 - q^{d(1-z)} = 0$ but $\frac{\mathrm{d}}{\mathrm{d}z}(1 - q^{d(1-z)}) = 1 + q^{d(1-z)}\log(q)d \neq 0$ at the point $z = 1$. Uniqueness of the continuation follows from the fact that $\mathbb{C} \setminus \{0,1\}$ is connected and ζ_C is already defined uniquely on the open set $\{z \in \mathbb{C} \mid \Re(z) > 1\}$.

Furthermore, the series

$$\sum_{k=0}^{\infty} N_C(kP)^{-z} = \sum_{k=0}^{\infty} N_C(P)^{-kz} = \frac{1}{1 - N_C(P)^{-z}}$$

converges absolutely for $\Re(z) > 1$ for any prime divisor P. The series

$$\sum_{P \in \mathrm{Div}(C) \text{ prime}} \sum_{k=0}^{\infty} N_C(kP)^{-z}$$

is a subseries of

$$\sum_{D \in \mathrm{Div}(C)} N_C(D)^{-z}$$

and must thus also converge absolutely for $\Re(z) > 1$. Hence, the product

$$\prod_{P \in \mathrm{Div}(C) \text{ prime}} \frac{1}{1 - N_C(P)^{-z}}$$

converges absolutely as well and the identity

$$\sum_{D\in\mathrm{Div}(C)} N_C(D)^{-z} = \prod_{P\in\mathrm{Div}(C)\text{ prime}} \sum_{k=0}^{\infty} N_C(kP)^{-z}$$
$$= \prod_{P\in\mathrm{Div}(C)\text{ prime}} \frac{1}{1-N_C(P)^{-z}}$$

follows by multiplying out the product. □

Theorem 2.2.6 ([Sch31, §8.2b)]). *There is a divisor of degree 1.*

Proof. Assume the degrees of all divisors are multiples of $d > 1$. Consider the (unique) field extension K/k of degree d together with the curve $C_K := C \times_k K$. Every prime divisor P of C splits into exactly d conjugate divisors $P'_1, \ldots, P'_d$ over K that all have degree $\frac{\deg(P)}{d}$ and therefore norm $N_{C_K}(P'_i) = |K|^{\deg(P)/d} = q^{\deg(P)} = N_C(P)$. That means that

$$\zeta_{C_K}(z) = \prod_{P'\in\mathrm{Div}(C_K)\text{ prime}} \frac{1}{1-N_{C_K}(P')^{-z}}$$
$$= \left(\prod_{P\in\mathrm{Div}(C)\text{ prime}} \frac{1}{1-N_C(P)^{-z}}\right)^d = \zeta_C(z)^d$$

for all $z \in \mathbb{C}$ satisfying $\Re(z) > 1$. But then, the analytic continuations of $\zeta_{C_K}(z)$ and $\zeta_C(z)^d$ have to coincide. But since $\zeta_{C_K}(z)$ has a pole of order 1 at $z = 0$ and $\zeta_C(z)^d$ has a pole of order d at $z = 0$, this means that $d = 1$. □

Remark 2.2.7. This can also be seen from the Hasse-Weil bound: The number of $\mathbb{F}_{q^n}$-points $N(n)$ on the curve C satisfies the inequality $|N(n) - (q^n + 1)| \leq 2g\sqrt{q^n}$, so $N(n) > 0$ for all sufficiently large n. In particular, there exists a divisor D of the $\mathbb{F}_q$-curve C of every sufficiently large degree (take a multiple of the sum of conjugates of a point). Hence, there exist two divisors whose degrees differ by 1.

2.3 Lang's Theorem for abelian varieties

In this section we are going to prove a broad generalization of the result from the previous section using arguments of a more geometrical nature.

See [Mil08] for an introduction to abelian varieties.

The following famous theorem has been proved in [Lan55]. The theorem is about certain projective varieties G over finite fields $k = \mathbb{F}_{p^n}$. Over finite fields, we have the Frobenius morphism $F : G \to G$ (see [Har77, page 301]). Its n-th power $F^n : G \to G$ is a $\operatorname{Spec} k$-morphism (since the n-th power of the Frobenius morphism on $\operatorname{Spec} k$ is the identity). Now, the variety G contains a point if and only if there is a point P on $G_{\overline{k}} = G \times_k \overline{k}$ such that $F^n(P) = P$.

Theorem 2.3.1 (Lang). *Let $k = \mathbb{F}_{p^n}$ be a finite field, $\overline{k}$ its algebraic closure and let G be a variety such that $G_{\overline{k}}$ can be made an abelian variety by choosing an appropriate group structure. Then the morphism $f : G_{\overline{k}} \to G_{\overline{k}}$ given by $P \mapsto F^n(P) - P$ is surjective. In particular the image contains the point 0 (the neutral element of the group operation), so G contains a point $x_0 \in G(k)$ defined over k. If $0 \in G(k)$, the group structure is already defined over the base field k.*

Proof. It is a well-known fact (cf. [Mil08, Cor. 1.2]) that every morphism of schemes between abelian varieties is a group homomorphism up to some translation, i.e., there exists a homomorphism $g : G_{\overline{k}} \to G_{\overline{k}}$ and a point $Q \in G_{\overline{k}}(\overline{k})$ such that $F^n(P) - P = f(P) = g(P) + Q$ for all points $P \in G_{\overline{k}}(\overline{k})$. Assume f (and hence g) is not surjective. For dimension reasons (cf. [Mil08, Prop. 7.1]), this means that the kernel of the group homomorphism $g : G_{\overline{k}}(\overline{k}) \to G_{\overline{k}}(\overline{k})$ has to be infinite.

Now, for any point $P \in \ker(g) \subseteq G_{\overline{k}}(\overline{k})$, the value $h(m) := F^{nm}(P) - P$ satisfies $h(0) = 0$ and the recurrence relation

$$\begin{aligned} h(m+1) &= F^n(F^{nm}(P)) - P = g(F^{nm}(P)) + Q + F^{nm}(P) - P \\ &= g(h(m) + P) + Q + h(m) = g(h(m)) + Q + h(m). \end{aligned}$$

A point $P \in \ker(g) \subseteq G_{\overline{k}}(\overline{k})$ lies in $G(\mathbb{F}_{p^{nm}})$ if and only if $h(m) = 0$. But since h is independent of the point P (according to the recurrence relation), this implies that $\ker(g) \subseteq G(\mathbb{F}_{p^{nm}})$ for some m (as $\overline{k} = \bigcup_m \mathbb{F}_{p^{nm}}$). However, this is absurd because $G(\mathbb{F}_{p^{nm}})$ is finite while $\ker(g)$ is infinite. Hence, f is surjective.

Assume now that $0 \in G(k)$. Then, the morphism $G_{\overline{k}} \times G_{\overline{k}} \to G_{\overline{k}}$ defined by $(P, Q) \mapsto F^{-n}(F^n(P) + F^n(Q))$ defines a group structure on $G_{\overline{k}}$ with neutral element 0. But since the group structure on $G_{\overline{k}}$ is uniquely determined by the choice of the neutral element (cf. [Mil08, Prop. Remark 1.3]), this must define exactly the original group structure. That means that the group structure on $G_{\overline{k}}$ is defined over the base field k (since it is invariant under F^n). ☐

Corollary 2.3.2. *Let $k = \mathbb{F}_{p^n}$ be a finite field and G an abelian variety over k. Let X be a principal homogeneous space over G (i.e., a non-empty variety over k together with a morphism $m : G \times X \to X$ such that $m_x : G_{\overline{k}} \to X_{\overline{k}}$ defined by $g \mapsto m(g, x)$ is an isomorphism for all $x \in X(\overline{k})$). Then, X is trivial, i.e., it contains a k-point: $X(k) \neq \emptyset$*

Proof. Take any point $x \in X_{\overline{k}}$. The group structure on $G_{\overline{k}}$ carries over to $X_{\overline{k}}$ via the isomorphism $m_x : G_{\overline{k}} \to X_{\overline{k}}$. Lang's theorem then shows that $X(k) \neq \emptyset$. ☐

We now want to use this corollary to show that $\mathrm{Pic}^d(C)$ is non-empty for a projective curve over a finite field k. To do this, we construct an abelian variety $J(C)$ parametrizing $\mathrm{Pic}^0(C)$ and obtain $\mathrm{Pic}^d(C)$ as a principal homogeneous space over $J(C)$.

To any smooth projective curve C over a field k, one can associate a variety $J(C)$, the *Jacobian variety of C*, with the fundamental property that its points correspond to elements of $\mathrm{Pic}^0(C)$ (at least when $C(k) \neq \emptyset$). See [Mil08, Chapter III] for an introduction to Jacobians.

We will only roughly sketch the construction of $J(C)$, here. Let g be the genus of the curve C. Using Galois descent (cf. [Mil12, Chapter 16]), it can be shown that it is sufficient to construct $J(C_{k'})$ for a finite extension $k'|k$. According to the Riemann-Roch theorem, every divisor class d of sufficiently large degree r contains an effective divisor. Let $r = 2g+1$, so that we have $l(d) = \deg(D) - g + 1 = g + 2$. Let us consider the symmetric product $\mathrm{Sym}^r(C) = C^r/S_r$. Points on $\mathrm{Sym}^r(C)$ correspond to unordered r-tuples of points on $C_{\overline{k}}$ and hence to effective divisors of degree r. We would now like to construct a closed subvariety of $\mathrm{Sym}^r(C)$ containing exactly one representative D of each divisor class $d \in \mathrm{Pic}^{2g+1}(C)$. As it turns out, this can be done at least locally by considering only divisors $D \geq D'$ for some fixed effective divisor D' of degree $g+1$ (over a finite extension of k). We can then glue together the resulting subvarieties of $\mathrm{Sym}^r(C)$ to obtain a variety $J(C)$ representing $\mathrm{Pic}^{2g+1}(C)$. Over a finite extension of k, there is at least one $d_0 \in \mathrm{Pic}^{2g+1}(C)$ and we can in fact identify the points of $J(C)$ with the elements of $\mathrm{Pic}^0(C)$ via the map $d \mapsto d - d_0$.

There is also an analytic way to construct $J(C)$ for curves C defined over the field $\mathbb{C}$ of complex numbers (cf. [HS00, Section A.6]): choose a basis $\gamma_1, \dots, \gamma_{2g}$ of the homology group $H_1(C, \mathbb{Z})$ and a basis $\omega_1, \dots, \omega_g$ of the vector space $H^0(C, K)$ of 1-forms on C. Then define

$$\Omega_i = \left(\int_{\gamma_i} \omega_1, \dots, \int_{\gamma_i} \omega_g \right) \qquad \text{for } 1 \leq i \leq 2g.$$

The $2g$ vectors $\Omega_i \in \mathbb{C}^g$ generate a lattice $\Lambda \subseteq \mathbb{C}^g$. One can write down a Riemann form with respect to the lattice Λ, so $J(C) = \mathbb{C}^g/\Lambda$ is an abelian variety. Choosing any point $p_0 \in C$, we obtain a map $C \to J(C)$ via

$$p \mapsto \left[\left(\int_\delta \omega_1, \dots, \int_\delta \omega_g \right) \right]$$

where δ is any path from p_0 to p. This map doesn't depend on the choice of the path δ since the vectors belonging to closed curves (linear combinations of $\gamma_1, \dots, \gamma_{2g}$) lie in the lattice Λ. This map can be linearly extended to a map $\mathrm{Div}(C) \to J(C)$ and it turns out that this induces an isomorphism $\mathrm{Pic}^0(C) \cong J(C)$ (which is also independent on the choice of p_0).

Theorem 2.3.3. *Let k be a finite field and C a smooth projective curve over k. Then, $\mathrm{Pic}^d(C)$ is nonempty for all integers d.*

Proof. Reviewing the algebraic construction of $J(C)$, it turns out that $\mathrm{Pic}^d(C)$ can also be turned into a variety which is a principal homogeneous space over $J(C)$ (and hence an abelian variety over $\overline{k}$). Therefore, $\mathrm{Pic}^d(C)^\Gamma \neq \emptyset$ due to Lang's theorem (and its corollary). Together with the following lemma, this shows that we in fact have $\mathrm{Pic}^d(C) \neq \emptyset$. □

Lemma 2.3.4. *If $k = \mathbb{F}_q$ is a finite field and $\Gamma = \mathrm{Gal}(\overline{k}, k)$ its absolute Galois group, then the isomorphism $\mathrm{Div}^d(C) \cong \mathrm{Div}^d(C_{\overline{k}})^\Gamma$ induces an isomorphism $\mathrm{Pic}^d(C) \cong \mathrm{Pic}^d(C_{\overline{k}})^\Gamma$.*

Proof. Injectivity holds for all projective varieties and surjectivity follows from the fact that the Brauer group of a finite field $k = \mathbb{F}_q$ is trivial. □

2.4 Local obstructions

Let X be a projective variety over the global field k.

Obviously, if $X(k_v) = \emptyset$ for some valuation v (or, equivalently, $X(\mathbb{A}_k) = \emptyset$), then $X(k) = \emptyset$. For example, the variety $X \subseteq \mathbb{P}^2_{\mathbb{Q}}$ given by $X_0^2 + X_1^2 + X_2^2 = 0$ has no $\mathbb{Q}$-point since it doesn't even have an $\mathbb{R} = \mathbb{Q}_\infty$-point.

If the converse holds, i.e., if $X(\mathbb{A}_k) \neq \emptyset$ implies $X(k) \neq \emptyset$, we say that the variety X fulfills the *Hasse principle*. For example, every quadric surface satisfies the Hasse principle. However, there are counterexamples to this principle: for example Selmer's curve $3X_0^3 + 4X_1^3 + 5X_2^3 = 0$ doesn't have a $\mathbb{Q}$-point, although it has an $\mathbb{R}$-point and a $\mathbb{Z}_p$-point for every prime p.

Let us now return to del Pezzo surfaces of degree 4. The following theorem shows that we often have to consider only the archimedian places and the places dividing 2.

Theorem 2.4.1. *Let $X \subseteq \mathbb{P}^n_{\mathcal{O}_v}$ be the intersection of two quadrics Q_∞, Q_0 over the valuation ring of the field k corresponding to some non-archimedian place $v \nmid 2$. Assume that the discriminant of the corresponding characteristic form $\chi(Q)$ is squarefree. Then, $X(\mathcal{O}_v) \neq \emptyset$.*

Proof. Let $\mathcal{O}_v/\mathfrak{m}_v \cong \mathbb{F}_q$. Since the discriminant of $\chi(Q)$ is squarefree, the form $\chi(Q)$ must have at least 4 distinct zeros over $\overline{\mathbb{F}_q}$. The following lemma then shows that $X \times_{\mathcal{O}_v} \mathbb{F}_q$ contains a nonsingular point. By Hensel's Lemma, nonsingular $\mathbb{F}_q$-points can be lifted to $\mathcal{O}_v$. □

Lemma 2.4.2. *Let $n \geq 4$ and let $X \subseteq \mathbb{P}^n_{\mathbb{F}_q}$ be the intersection of two quadrics Q_∞, Q_0 over a finite field $\mathbb{F}_q$ of odd characteristic and assume that the corresponding characteristic form $\chi(Q)$ has at least n distinct roots over $\overline{\mathbb{F}_q}$. Then, $X(\mathbb{F}_q)$ contains a non-singular point.*

Proof. If $\chi(Q)$ has $n+1$ distinct roots over $\overline{\mathbb{F}_q}$, then Lemma 1.3.3 shows that X is smooth. But $X(\mathbb{F}_q) \neq \emptyset$ since the Tsen rank of $\mathbb{F}_q$ is 1 (see Theorem 2.1.7).

Otherwise, let $(u : v)$ be the root of multiplicity 2 of the characteristic form. Since $\mathbb{F}_q$ is a perfect field and all other roots are simple, the root $(u : v)$ must be defined over k. By a change of variables (i.e., of the matrices Q_∞ and Q_0), we may thus assume that $(u : v) = (1 : 0)$. Again by changing variables, we may also assume that Q_∞ is a diagonal matrix. Since $\det(Q_\infty) = 0$, one of the entries must be 0, so we can assume that the matrix has the following form:

$$Q_\infty = \begin{pmatrix} a_1 & & & & \\ & a_2 & & & \\ & & \ddots & & \\ & & & a_n & \\ & & & & 0 \end{pmatrix}$$

From Lemma 1.3.1 we know that the dimension $\dim \ker(Q_\infty)$ must be either 1 or 2, so we can assume that $a_1, \ldots, a_{n-1} \neq 0$.

Let $Q_0 = (b_{ij})_{1 \le i,j \le n+1}$.

We now want to show that there exists some point $x \in \mathbb{P}^n_{\mathbb{F}_q}$ such that

$$0 = \mathfrak{q}_\infty(x) = \sum_{i \le n} a_i x_i^2$$

and

$$0 = \mathfrak{q}_0(x) = \sum_{i,j} b_{ij} x_i x_j$$

but

$$x \notin \ker(Q_\infty)$$

since it must then be a nonsingular point of $X(\mathbb{F}_q)$.

We already know that the two equations have a common solution, so we may assume that there is some $y \in \mathbb{P}^n_{\mathbb{F}_q}$ such that $\mathfrak{q}_\infty(y) = \mathfrak{q}_0(y) = 0$ and $Q_\infty y = 0$. Changing basis on $\ker(Q_\infty)$ again (which is spanned by the last one or two coordinate vectors), we can assume that y is the last coordinate vector. Then, since $\mathfrak{q}_0(y) = 0$, we have $b_{n+1,n+1} = 0$. But that means that the condition

$$0 = \mathfrak{q}_0(x) = x_{n+1} \sum_{i \le n} 2 b_{n+1,i} x_i + \sum_{i,j \le n} b_{ij} x_i x_j$$

is linear in x_{n+1} when we fix all other x_i. We can solve this equation for x_{n+1} whenever $\sum_{i \le n} b_{n+1,i} x_i \neq 0$ (for example if this sum is 1). If all $b_{n+1,i} = 0$ for $i \le n$, then $(0, \ldots, 0, 1)$ would lie in the kernel of both Q_∞ and Q_0, so $\chi(Q)$ would be constantly zero. Otherwise, we can assume $b_{n+1,3} \neq 0$. Solving the equation $\sum_{i \le n} b_{n+1,i} x_i = 1$ for x_3 and putting that into the equation $\mathfrak{q}_\infty(x) = 0$ and also setting $x_4, \ldots, x_n = 0$, we are left with one quadratic equation in the variables x_1 and x_2:

$$0 = a_1 x_1^2 + a_2 x_2^2 + a_3 b_{n+1,3}^{-2} (1 - b_{n+1,1} x_1 - b_{n+1,2} x_2)^2$$

See Lemma 2.4.3 for the (tedious) proof that such an equation always has a solution. Then,

$$\left(x_1, x_2, b_{n+1,3}^{-1}(1 - b_{n+1,1} x_1 - b_{n+1,2} x_2), 0, \ldots, 0, -\frac{1}{2} \sum_{i,j \le n} b_{ij} x_i x_j \right)$$

is a nonsingular point of $X(\mathbb{F}_q)$. □

Lemma 2.4.3. *Let $\mathbb{F}_q$ be a finite field of odd characteristic and let $a_1, a_2, a_3, b_1, b_2 \in \mathbb{F}_q$ with $a_1, a_2, a_3 \neq 0$. Then, the equation*

$$a_1x_1^2 + a_2x_2^2 + a_3(1 + b_1x_1 + b_2x_2)^2 = 0 \tag{2.1}$$

has a solution $(x_1, x_2) \in \mathbb{F}_q^2$.

Proof. We want to find a solution to the equation

$$(a_1 + a_3b_1^2)x_1^2 + (a_2 + a_3b_2^2)x_2^2 + 2a_3b_1b_2x_1x_2 + 2a_3b_1x_1 + 2a_3b_2x_2 + a_3 = 0.$$

Let us first consider a few special cases:
If $a_1 + a_3b_1^2 = 0$ (in particular $b_1 \neq 0$), then the equation is equivalent to

$$(a_2 + a_3b_2^2)x_2^2 + 2a_3b_1(b_2x_2 + 1)x_1 + 2a_3b_2x_2 + a_3 = 0$$

and $\left(-\frac{1}{2b_1}, 0\right)$ is a solution.
We can therefore assume that $a_1 + a_3b_1^2 \neq 0$ and similarly $a_2 + a_3b_2^2 \neq 0$.
If $(a_1 + a_3b_1^2)(a_2 + a_3b_2^2) = (a_3b_1b_2)^2$ (in particular $a_3b_1b_2 \neq 0$), then the equation is equivalent to

$$(a_1 + a_3b_1^2)\left(x_1 + \frac{a_2 + a_3b_2^2}{a_3b_1b_2}x_2\right)^2 + 2a_3(b_1x_1 + b_2x_2) + a_3 = 0.$$

But since $\frac{a_2+a_3b_2^2}{a_3b_1b_2} = \frac{b_2}{b_1} + \frac{a_2}{a_3b_1b_2} \neq \frac{b_2}{b_1}$, the system of linear equations

$$\begin{aligned} x_1 + \frac{a_2 + a_3b_2^2}{a_3b_1b_2}x_2 &= 0 \\ b_1x_1 + b_2x_2 &= -\frac{1}{2} \end{aligned}$$

must have a solution which is then also a solution to the original equation.
We can therefore also assume that $(a_1 + a_3b_1^2)(a_2 + a_3b_2^2) \neq (a_3b_1b_2)^2$. Then, the equation is equivalent to

$$\begin{aligned} &(a_1 + a_3b_1^2)\left(x_1 + \frac{a_3b_1b_2}{a_1 + a_3b_1^2}x_2 + \frac{a_3b_1}{a_1 + a_3b_1^2}\right)^2 \\ &+ \frac{(a_1 + a_3b_1^2)(a_2 + a_3b_2^2) - (a_3b_1b_2)^2}{a_1 + a_3b_1^2}\left(x_2 + \frac{a_1a_3b_2}{(a_1 + a_3b_1^2)(a_2 + a_3b_2^2) - (a_3b_1b_2)^2}\right)^2 \\ &+ \frac{a_1a_2a_3}{(a_1 + a_3b_1^2)(a_2 + a_3b_2^2) - (a_3b_1b_2)^2} \\ &= 0. \end{aligned}$$

This equation is of the form

$$c_1y_1^2 + c_2y_2^2 + c_3 = 0$$

for some $c_1, c_2, c_3 \in \mathbb{F}_q$ with $c_1, c_2 \neq 0$ and $x_2 = y_2 - d_1$ and $x_1 = y_1 - d_2x_2 - d_3$ for some $d_1, d_2, d_3 \in \mathbb{F}_q$. But such an equation has a solution (y_1, y_2) since the sets $\{c_1y_1^2 \mid y_1 \in \mathbb{F}_q\}$ and $\{-c_2y_2^2 - c_3 \mid y_2 \in \mathbb{F}_q\}$ each have $\frac{q+1}{2} > \frac{q}{2}$ elements and must therefore have at least one element in common. □

2.5 Brauer–Manin obstructions

Let again X be a projective variety over the global field k.

Failures of the Hasse principle can sometimes be explained by Brauer–Manin obstructions, which exhibit a certain incompatibility between the local solutions. We will just give a short introduction to the theory of Brauer–Manin obstructions. The variety X has an associated *Brauer group* $\mathrm{Br}(X)$. Each element $A \in \mathrm{Br}(X)$ comes with a map $A_v : X(k_v) \to \mathbb{Q}/\mathbb{Z}$ for each valuation v. Importantly, each such element $A \in \mathrm{Br}(X)$ and each point $x \in X(k)$ satisfies the identity $\sum_v A_v(x) = 0$ where the sum contains only finitely many nonzero terms. Let us write $X(\mathbb{A}_k)^{\mathrm{Br}(X)}$ for the set of points $x \in X(\mathbb{A}_k)$ such that almost all $A_v(x_v)$ are zero and furthermore $\sum_v A_v(x_v)$ for all $A \in \mathrm{Br}(X)$. It is then obvious that $X(k) \subseteq X(\mathbb{A}_k)^{\mathrm{Br}(X)}$.

Let now $X \subseteq \mathbb{P}^4_k$ be a del Pezzo surface of degree 4 over a global field k defined by the pencil Q of quadrics. Assume that X has solutions over every completion k_v. For simplicity, assume that Q_∞ is invertible, so that the polynomial $\chi(Q)(u:1)$ has 5 distinct roots $r_1, \ldots, r_5 \in \overline{k}$. Furthermore, choose $\varepsilon_i \in \varepsilon(Q_{r_i}) \in k(r_i)^\times / k(r_i)^{\times 2}$ (so in particular $\varepsilon_i \in k(r_i)^\times$) so that for any automorphism σ of $k(r_1, \ldots, r_5)$ with $\sigma(r_i) = r_j$, we also have $\sigma(\varepsilon_i) = \varepsilon_j$. Also choose a linear form t_i defining the tangent space to a smooth point on Q_{r_i} and assume that $\sigma_i(t_i) = t_j$ whenever $\sigma(r_i) = r_j$. Then, the Brauer group $\mathrm{Br}(X)$ can be explicitly computed depending on the roots r_i, the discriminants ε_i an the linear forms t_i (see [VAV12, Theorem 3.4] and [Sch13, Theorem 2.18]). The computation is closely linked with the action of the Galois group $\mathrm{Gal}(L|k)$ on the Picard group $\mathrm{Pic}(X_{\overline{k}})$.

Definition 2.5.1. We call a pair (r_1, r_2) of roots of $\chi(Q)$ a *nice pair* if

a) $r_1, r_2 \in \mathbb{P}^1_k$ and $\varepsilon_1 = \varepsilon_2 \neq 1 \in k^\times / k^{\times 2}$ or

b) r_1 and r_2 are the zeroes of an irreducible quadratic factor of $\chi(Q)$ and $\varepsilon_1 = \varepsilon_2 \neq 1 \in k(r_1)^\times / k(r_1)^{\times 2}$ but $\varepsilon_1 \cdot \varepsilon_2 = 1 \in k^\times / k^{\times 2}$.

Theorem 2.5.2. *Exactly one of the following three statements holds:*

a) There is no nice pair. Then, $\mathrm{Br}(X)/\mathrm{Br}(k) = 0$.

b) There is exactly one nice pair (r_1, r_2)*. Then,* $\mathrm{Br}(X)/\mathrm{Br}(k) = \left\{0, \left(\varepsilon(Q_{r_1}), \frac{t_1}{t_2}\right)\right\}$ *has 2 elements.*

c) There are three roots r_1, r_2, r_3 *of* $\chi(Q)$ *such that* (r_1, r_2)*,* (r_1, r_3) *and* (r_2, r_3) *are nice pairs. Then,* $\mathrm{Br}(X)/\mathrm{Br}(k) = \left\{0, \left(\varepsilon(Q_{r_1}), \frac{t_1}{t_2}\right), \left(\varepsilon(Q_{r_1}), \frac{t_1}{t_3}\right), \left(\varepsilon(Q_{r_2}), \frac{t_2}{t_3}\right)\right\}$ *has 4 elements.*

The notation $A = (a, b)$ *refers to a quaternion algebra. The corresponding maps* $A_v : X(k_v) \to \mathbb{Q}/\mathbb{Z}$ *are simply given by the Hilbert symbol:*

$$A_v(x) = \begin{cases} [0], & \text{the equation } a(x)\lambda^2 + b(x)\mu^2 = 1 \text{ has a solution in } k_v \\ [1/2], & \text{otherwise} \end{cases}$$

In [VAV12, Remark 1.4], Várilly-Alvarado and Viray gave the following way to check whether $X(\mathbb{A}_k)^{\mathrm{Br}(X)} = \emptyset$.

Theorem 2.5.3. *The set $X(\mathbb{A}_k)^{\mathrm{Br}(X)}$ of adelic points compatible with the Brauer group is empty if and only if there exists a subspace $W \subseteq \mathbb{P}_k^4$ of dimension $1 \leq n \leq 3$ such that for all subspaces $W \subseteq V \subseteq \mathbb{P}_k^4$ of dimension $n+1$ the set $(X \cap V)(\mathbb{A}_k)$ is empty.*

Remark 2.5.4. It is obvious that the existence of such a subspace W implies $X(k) = \emptyset$.

It is a conjecture by Colliot-Thélène and Sansuc that each failure of the Hasse principle on del Pezzo surfaces of degree 4 is explained by Brauer–Manin obstructions. The conjecture has been proved for many classes of del Pezzo surfaces of degree 4 under the assumption that Schinzel's hypothesis is true and the Tate-Shafarevich groups of elliptic curves are finite. Also see [VAV12, Section 5.5] for an overview of available results.

3 Arakelov theory

We only give a very brief introduction to Arakelov theory following [Lan88] and [dJ04].

When studying a projective variety X, it is often helpful to look at the group of divisors of X (or, more generally, the group of algebraic cycles) and at the intersection product defined on this space. To obtain a sensible intersection product (which is invariant under rational equivalence), it is crucial that the variety X is complete.

In arithmetic applications this is sometimes not the case: rather, one would like to study some scheme X over $\operatorname{Spec}\mathcal{O}_k$ where $\mathcal{O}_k$ is the ring of integers of some number field. But the base scheme $\operatorname{Spec}\mathcal{O}_k$ and the scheme X are (usually) not complete!

Arakelov theory provides a way to overcome this problem and again be able to define a useful intersection product. Essentially, we add the archimedian valuations (places at infinity) to $B = \operatorname{Spec}\mathcal{O}_k$ to make this scheme complete. More precisely, we add a prime Arakelov divisor for each archimedian valuation and allow real coefficients for this prime divisor.

Arakelov divisors of $B = \operatorname{Spec}\mathcal{O}_k$. We call a finite formal sum

$$\sum_{v\nmid\infty} a_v v + \sum_{v\mid\infty} b_v v$$

with $a_v \in \mathbb{Z}$ and $b_v \in \mathbb{R}$ an *Arakelov divisor* of B. Its degree is $\sum_{v\nmid\infty} a_v \deg(v) + \sum_{v\mid\infty} b_v \deg(v)$. We obtain the group of Arakelov divisors

$$\operatorname{Div}_{\mathrm{Ar}}(B) = \bigoplus_{v\nmid\infty} \mathbb{Z}\cdot v \oplus \bigoplus_{v\mid\infty} \mathbb{R}\cdot v.$$

Note that the Arakelov divisors with $b_v = 0$ are exactly the divisors of $\operatorname{Spec}(\mathcal{O}_k)$. For an element $f \in k^\times$, we can construct the *principal divisor*

$$(f) = \sum_v v(f)v,$$

which has degree 0 according to the product formula of valuations.

As usually, we get the Arakelov class group $\operatorname{Pic}_{\mathrm{Ar}}(B)$, the group $\operatorname{Div}^0_{\mathrm{Ar}}(B)$ of degree 0 divisors and the degree 0 class group $\operatorname{Pic}^0_{\mathrm{Ar}}(B)$.

Remark 3.0.5. The group $\operatorname{Pic}^0_{\mathrm{Ar}}(B)$ is compact due to the finiteness of the ideal class group and Dirichlet's unit theorem.

Arakelov divisors on X. Now, we would like to define Arakelov divisors on an integral regular proper scheme X over $B = \operatorname{Spec} \mathcal{O}_k$, the generic fiber X_η of which is smooth. Although it is possible to carry out Arakelov theory in higher dimensions (cf. [GS92]), let us for simplicity assume that the generic fiber X_η is of dimension 1. We also assume that the genus g is not zero.

The prime divisors (in the usual sense) of X come in two categories: there are the *vertical divisors* whose image in B is finite and the *horizontal divisors* whose image in B is dense. Vertical divisors are irreducible components of fibers of B while horizontal divisors correspond to divisors of the generic fiber X_η.

To the set of vertical divisors, we now add one prime divisor X_v per archimedian valuation v, just as when defining the group of divisors on B. Such a valuation comes from an embedding $\sigma : k \hookrightarrow \mathbb{C}$ and we can thus also regard X_v as a smooth variety over $\mathbb{C}$ (in fact, a compact Riemann surface), by changing base on the generic fiber X_η. An Arakelov divisor of X is a finite formal sum

$$A + \sum_{v|\infty} b_v X_v$$

where $A \in \operatorname{Div}(X)$ is a divisor in the usual sense and $b_v \in \mathbb{R}$.

We now define a volume form on each fiber X_v at infinity. We take an orthonormal basis $\omega_1, \ldots, \omega_g \in H^0(X_v, K)$ of the space of 1-forms with respect to the Hermitian inner product $\langle \varphi, \psi \rangle = \frac{i}{2} \int_X \varphi \wedge \psi$. Then, we obtain the volume form $\mu_v = \frac{i}{2g} \sum_{t=1}^{g} \omega_t \wedge \overline{\omega_t}$. The principal divisor associated to an element $f \in K(X)^\times$ is defined as

$$(f) = (f)_{\text{fin}} + \sum_{v|\infty} \left(\int_X v(f) \mu_v \right) X_v$$

where $(f)_{\text{fin}}$ is the usual (finite) principal divisor.

Next, we would like to define an intersection product for the (Arakelov) divisors of X. As in the case of complete varieties, we first do this for distinct prime divisors D and E and then use bilinearity and a "moving lemma" to extend it to all divisors.

First, let the total intersection number be $(D.E) = \sum_v (D.E)_v$ where we define the local intersection number $(D.E)_v = (E.D)_v$ separately for each valuation v.

1. If v is finite and one of the divisors D and E is at infinity, we let $(D.E)_v = 0$. We also use this definition in case $D = E$ are the same divisor at infinity.
2. If v is finite and both D and E are finite divisors, we let

$$(D.E)_v = \sum_{x \in X_v} \operatorname{length}_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/(f_x, g_x)$$

   if D and E are described by the functions f_x and g_x around the point x. This matches the usual definition of the intersection product.
3. If v is infinite and both D and E are vertical, we let $(D.E)_v = 0$. We also use this definition in case $D = E$ are the same divisor at infinity.

4. If v is infinite and D is a vertical divisor but $D \neq X_v$, we let $(D.E)_v = 0$.
5. If v is infinite and $D = X_v$ and E is horizontal, we let $(D.E)_v = \deg(f) = [k(E) : k]$.
6. If v is infinite and both D and E are horizontal, we let $(D.E)_v = -\log(G_v(D, E))$. Here, D and E are interpreted as divisors on X_v and $G_v : X_v \times X_v \to \mathbb{R}_{\geq 0}$ is the so-called Arakelov–Green function associated to μ_v.

It can be shown that every divisor D has a rationally equivalent divisor D' so that the supports of D and D' share at most fibers at infinity (the moving lemma) and that the intersection product doesn't depend on the rational equivalence class we chose (as long as it is defined)[1], so we can extend the intersection product to all pairs of Arakelov divisors of X.

Many important theorems of intersection theory have analogs in Arakelov theory: for example the Hodge index theorem, the adjunction formula for canonical sheaves and the Riemann-Roch theorem.

[1]This follows from the definition of the Arakelov–Green function G_v associated to μ_v.

4 Del Pezzo surfaces over $\mathbb{Q}$

Let X be a del Pezzo surface of degree 4 over $\mathbb{Q}$ associated to the pencil Q of quadrics. Again, assume that Q_∞ is invertible and recall the definition of $r_1, \dots, r_5$ and $\varepsilon_1, \dots, \varepsilon_5$ from section 2.5.

We are now interested in the field extension $L|K$ where $K = \mathbb{Q}(r_1, \dots, r_5)$ and $L = K(\sqrt{\varepsilon_1}, \dots, \sqrt{\varepsilon_5})$.

When looking at the Brauer group of X (cf. Theorem 2.5.2), it turns out that it vanishes when the Galois group $\mathrm{Gal}(K|\mathbb{Q})$ is large. For example, if $\chi(Q)$ is irreducible, then $\mathrm{Br}(X) = 0$. A large Galois group $\mathrm{Gal}(L|\mathbb{Q})$ can also be helpful, as it then describes most automorphisms of $\mathrm{Pic}(X_{\overline{k}})$.

Example 4.0.6. Let a, b, c be non-zero rationals such that $a^2 + 2b^2 = c^2$ (for example $a = 1, b = 2, c = 3$) and let

$$Q_\infty = \begin{pmatrix} 1 &&&& \\ & -1 &&& \\ && 1 && \\ &&& -1 & \\ &&&& 1 \end{pmatrix} \quad Q_0 = \begin{pmatrix} -c &&&& \\ & -a &&& \\ && 0 && \\ &&& a & \\ &&&& c \end{pmatrix}.$$

Then, $\{r_1, r_2, r_3, r_4, r_5\} = \{0, -a, a, -c, c\}$ and all $\varepsilon(Q_{r_i})$ are trivial in $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$, so $L = K = \mathbb{Q}$. For any $(x, y, z) \in \mathbb{Q}^3 \setminus \{0\}$ with $2x^2 - 2y^2 + z^2 = 0$ (for example $x = 1, y = 3, z = 4$), the point $(x : y : z : y : x)$ lies on X.

Due to Lemma 1.3.5, we can simultaneously diagonalize Q_∞ and Q_0 over K and then Lemma 1.3.7 gives us formulas for the roots r_1 and the discriminants $\varepsilon_1, \dots, \varepsilon_5$. First of all, it is clear from these formulas that $\prod_i \varepsilon_i \in K^{\times 2}$ is a square. That means that the Galois group $\mathrm{Gal}(L|K)$ is a subgroup of $(\mathbb{Z}/2\mathbb{Z})^4$.

Lemma 4.0.7. *Let $H \subseteq S_5$ be a transitive subgroup (i.e., for all $i \in \{1, \dots, 5\}$ there is some $\sigma \in H$ such that $\sigma(1) = i$). Then, H contains a permutation of order 5.*

Proof. The subgroup $h = \{\sigma \in H \mid \sigma(1) = 1\}$ has exactly 5 left cosets because H is transitive. Hence, the order of H is divisible by 5, so, by the Sylow theorems, H contains an element of order 5. □

We will need the following classical lemma about compositions of quadratic extensions (see for example [Bor08]).

Lemma 4.0.8. *Let K be a field of characteristic different from 2 and let $d_1, \dots, d_n \in K^\times$. Let $L = K[\sqrt{d_1}, \dots, \sqrt{d_n}]$. The extension $L|K$ has degree 2^n if and only if the number $\prod_{i \in S} d_i$ is not a square in K for any $\emptyset \neq S \subseteq \{1, \dots, n\}$.*

Proof. Assume $\prod_{i \in S} d_i$ is a square (with $\emptyset \neq S \subseteq \{1, \dots, n\}$). Let without limit of generality $1 \in S$. Then, $L|K$ obviously has degree at most 2^{n-1}, as $\sqrt{d_1} \in K[\sqrt{d_2}, \dots, \sqrt{d_n}]$.

For the converse, we use induction on n.

For $n = 1$, the claim is obvious.

For $n = 2$, we have to show that $\sqrt{d_2} \notin K[\sqrt{d_1}]$. Assume $x, y \in K$ such that $d_2 = (x + y\sqrt{d_1})^2 = x^2 + 2xy\sqrt{d_1} + y^2 d_1$. As $\sqrt{d_1} \notin K$ and $2 \neq 0$, this would imply $x = 0$ or $y = 0$, so $d_2 = y^2 d_1$ or $d_2 = x^2$ which are both impossible due to the assumption.

For $n > 2$, let $L' = K[\sqrt{d_1}, \dots, \sqrt{d_{n-2}}]$. We use the induction hypothesis for $n-2$ to see that $[L' : K] = 2^{n-2}$. According to the induction hypothesis for $n - 1$, we furthermore have $[L'[\sqrt{d_{n-1}}] : K] = [L'[\sqrt{d_n}] : K] = [L'[\sqrt{d_{n-1}}\sqrt{d_n}] : K] = 2^{n-1}$, so $\sqrt{d_{n-1}}, \sqrt{d_n}$ and $\sqrt{d_{n-1}}\sqrt{d_n}$ are no squares in L'. We can hence apply the induction hypothesis to the extension $L = L'[\sqrt{d_{n-1}}, \sqrt{d_n}]|L'$ and get that $[L : L'] = 4$. Together with $[L' : K] = 2^{n-2}$, this yields the claim. □

Lemma 4.0.9. *Assume that the characteristic form $\chi(Q)$ is irreducible and let again $K = \mathbb{Q}(r_1, \dots, r_5)$ and $L = K(\sqrt{\varepsilon_1}, \dots, \sqrt{\varepsilon_5})$. Then,*

$$\mathrm{Gal}(L|K) \cong \begin{cases} 0, & \varepsilon_1 \in K^{\times 2} \\ (\mathbb{Z}/2\mathbb{Z})^4, & \varepsilon_1 \notin K^{\times 2} \end{cases}$$

Proof. Since the characteristic form is irreducible, the group $\mathrm{Gal}(K|\mathbb{Q})$ acts transitively on $r_1, \dots, r_5$. Let $\sigma_i \in \mathrm{Gal}(K|\mathbb{Q})$ be such that $\sigma_i(r_1) = r_i$.

If $\varepsilon_1 \in K^{\times 2}$, then we even have $\varepsilon_i = \sigma_i(\varepsilon_1) \in K^{\times 2}$ for all i. It obviously follows, that $L = K$.

Otherwise, we have $\varepsilon_i = \sigma_i(\varepsilon_1) \notin K^{\times 2}$ for all i. Due to $\prod_{i=1}^5 \varepsilon_i \in K^{\times 2}$, we know that $L = K(\sqrt{\varepsilon_1}, \dots, \sqrt{\varepsilon_4})$. Assume $\mathrm{Gal}(L|K) \ncong (\mathbb{Z}/2\mathbb{Z})^4$. From Lemma 4.0.8 follows that there is some $\emptyset \neq S \subsetneq \{1, \dots, 5\}$ such that the product $\prod_{i \in S} \varepsilon_i$ is a square in $K^\times$. Due to $\prod_{i=1}^5 \varepsilon_i \in K^{\times 2}$, we may assume $|S| \leq 2$. If $|S| = 2$, we can use Lemma 4.0.7: There has to exist some $\tau \in \mathrm{Gal}(K|\mathbb{Q})$ of order 5. It is now easy to see that S is disjoint from τS or $\tau^2 S$ (the automorphism τ permutes the integers $1, \dots, 5$ when we identify them with the corresponding roots $r_1, \dots, r_5$). Let without limit of generality S be disjoint from τS. Hence, $|S \cup \tau S| = 4$. But now $\prod_{i \in S \cup \tau S} \varepsilon_i = (\prod_{i \in S} \varepsilon_i)\tau(\prod_{i \in S} \varepsilon_i) \in K^{\times 2}$ and, using $\prod_{i=1}^5 \varepsilon_i \in K^{\times 2}$ again, we therefore only have to consider the case $|S| = 1$. But then, $\prod_{i \in S} \varepsilon_i \in K^{\times 2}$ is impossible by assumption (since $\varepsilon_i \notin K^{\times 2}$ for all i). □

Theorem 4.0.10. *Assume one of the following:*

a) The matrix Q_∞ is anisotropic and has signature $(5, 0), (4, 1), (1, 4)$ or $(0, 5)$ and $\chi(Q)$ has 5 real roots and $\mathrm{Gal}(K|\mathbb{Q}) \cong S_5$.

b) The matrix Q_∞ is positive definite (signature $(5,0)$) or negative definite (signature $(0,5)$) and $\chi(Q)$ has exactly 3 real roots and $\operatorname{Gal}(K|\mathbb{Q}) \cong S_5$.

Then, the Galois group $\operatorname{Gal}(L|\mathbb{Q})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$.

Proof. We use the notation from Lemma 1.3.7. We then have

$$\varepsilon_i = \prod_{j\neq i}(a_i b_j - a_j b_i) = \prod_{j\neq i}(-a_i r_j a_j + a_j r_i a_i) = a_i^3 \left(\prod_j a_j\right)\left(\prod_{j\neq i}(r_i - r_j)\right).$$

If all roots $r_1, \ldots, r_5$ are real, say $r_1 < \cdots < r_5$, then $\prod_{j\neq i}(r_i - r_j)$ is positive for $i = 1, 3, 5$ and negative for $i = 2, 4$. Due to our assumption on the signature of Q_∞, this means that $\varepsilon_i < 0$ for some i. But then, ε_i is not a square in $\mathbb{Q}(r_1, \ldots, r_5) \subseteq \mathbb{R}$, so Lemma 4.0.9 shows the claim.

Otherwise, let $r_1 < r_2 < r_3$ be the real roots (so $r_5 = \overline{r_4}$). Assume without limit of generality that Q_∞ is positive definite. One can then easily see from the proof of Lemma 1.3.5, that $a_1, a_2, a_3 > 0$ and that we can arrange $a_5 = \overline{a_4}$. We then have $\varepsilon_1 > 0$. If the claim was false, then, according to Lemma 4.0.9, ε_1 would have to be a square in $K = \mathbb{Q}(r_1, \ldots, r_5)$. Hence, $\mathbb{Q}(r_1, \sqrt{\varepsilon_1})|\mathbb{Q}(r_1)$ would have to be a subextension of $\mathbb{Q}(r_1, \ldots, r_5)|\mathbb{Q}(r_1)$ of degree at most 2. But $\operatorname{Gal}(\mathbb{Q}(r_1, \ldots, r_5)|\mathbb{Q}(r_1)) \cong S_4$ has only one subgroup of index 2, namely A_4, so either $\mathbb{Q}(r_1, \sqrt{\varepsilon_1}) = \mathbb{Q}(r_1)$ or $\mathbb{Q}(r_1, \sqrt{\varepsilon_1})$ is the fixed field corresponding to A_4. Complex conjugation corresponds to the permutation $(4,5) \notin A_4$ of roots, so in the second case $\mathbb{Q}(r_1, \sqrt{\varepsilon_1})$ can't have a real embedding. But since $r_1 \in \mathbb{R}$ and $\varepsilon_1 > 0$, this is impossible. Hence, $\mathbb{Q}(r_1, \sqrt{\varepsilon_1}) = \mathbb{Q}(r_1)$, so $\sqrt{\varepsilon_1} \in \mathbb{Q}(r_1)$. Applying the automorphism of $\mathbb{Q}(r_1, \ldots, r_5)$ corresponding to the permutation $(1,2)$ of roots, we get $\sqrt{\varepsilon_2} \in \mathbb{Q}(r_2) \subseteq \mathbb{R}$. This yields a contradiction, since $\varepsilon_2 < 0$. □

Corollary 4.0.11. *Let Q_∞ be positive (or negative) definite and assume that $\chi(Q)$ is irreducible and has exactly 3 real roots. Then, the Galois group $\operatorname{Gal}(L|\mathbb{Q})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$.*

Proof. It is a classical fact that the Galois group of an irreducible polynomial $f \in \mathbb{Q}[X]$ of degree 5 with exactly 3 real roots is isomorphic to S_5: the Galois group is transitive, so by Lemma 4.0.7, it has to contain a 5-cycle. Furthermore, the Galois group has to contain a transposition corresponding to complex conjugation. But a 5-cycle and any transposition already generate the group S_5. □

Example 4.0.12. Let $Q_\infty = I_5$ be the identity matrix and

$$Q_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 \\ -1 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

The intersection of the quadrics

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0$$

$$2x_2x_3 - x_3^2 - 2x_2x_4 + 2x_3x_4 + x_4^2 - 2x_1x_5 - 2x_4x_5 = 0$$

defined by Q_∞ and Q_0 is a del Pezzo surface of degree 4. Its characteristic form is given by the irreducible polynomial $\chi(Q)(r:1) = \det(rQ_\infty + Q_0) = r^5 - 6r^3 - r^2 + 5r + 2$ that has 5 real roots. The Galois group is therefore $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$. The discriminant of the characteristic form is the prime number 81509. Of course, this del Pezzo surface doesn't have an $\mathbb{R}$-point, since $\mathbb{Q}_\infty$ is positive definite.

Example 4.0.13. Let

$$Q_\infty = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix} \qquad \text{and} \qquad Q_0 = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 & 0 \\ -1 & -1 & 0 & -1 & -1 \\ -1 & 1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

The corresponding del Pezzo surface has the characteristic form $\chi(Q)(r:1) = -r^5 + 2r^4 + 3r^3 - 5r^2 - r + 1$ that has 5 real roots. The Galois group is $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$. The discriminant of the characteristic form is the prime number 36497. The del Pezzo surface contains the point $(0,0,0,1,1)$, for example.

One can easily construct more examples using Theorem 1.5.1, of course.

5 Del Pezzo fibrations

From del Pezzo surfaces of degree 4 over fields k, we will now turn to del Pezzo surface fibrations of degree 4 over integral one-dimensional base schemes B. Let $k = K(B)$ be the function field of such a base scheme and assume $\mathrm{char}(k) \neq 2$.

We call a projective flat morphism $\pi : X \to B$ a *del Pezzo surface fibration of degree 4* if the fiber X_b is a del Pezzo surface of degree 4 over $\kappa(b)$ for almost each base point $b \in B$. We say that $\pi : X \to B$ has *square-free* discriminant if each fiber is a complete intersection of two quadrics with at worst a singularity of order 1 ([HT14]).

Let us apply some of our theory to the fibers X_b. First of all, each fiber X_b is defined by some pencil of quadrics Q^b and we can look at its characteristic form $\chi(Q^b)$. Consider for each $b \in B$ the corresponding zero set $D_b \subset \mathbb{P}^1$. These zero sets can be glued together to obtain a scheme D with a morphism $D \to B$ of degree 5. Each singular quadric Q_r^b has two families of planes, since we assumed that π has square-free discriminant. We obtain a double covering $\psi : \tilde{D} \to D$ parametrizing these families. The map $\psi : \tilde{D} \to D$ is étale. Let $\iota : \tilde{D} \to \tilde{D}$ be the map swapping the two families of planes over each point of D.

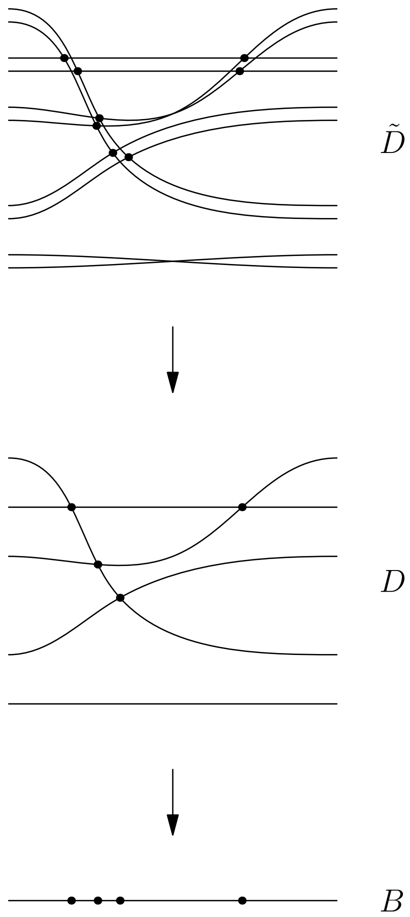
For sufficently nice base schemes, the covering induces a canonical map $\psi : J(\tilde{D}) \to J(D)$ between the Jacobians of these two curves and the morphism $1 - \iota$ induces a map $1 - \iota : J(\tilde{D}) \to J(\tilde{D})$. The *Prym variety* $\mathrm{Prym}(\tilde{D} \to D)$ is defined as the image of this map $1-\iota$. Obviously, $\psi \circ (1-\iota) = 0$, so we obtain the inclusion $\mathrm{Prym}(\tilde{D} \to D) \subseteq \ker(\psi)$ of groups.

Let us now try to construct a map u from the space S_π of sections of $\pi : X \to B$ to the Prym variety $\mathrm{Prym}(\tilde{D} \to D) \subseteq J(\tilde{D})$. To do this, fix some section $\sigma_0 : B \to X$ of π not passing through any singularity of a singular quadric in any fiber of π. Each point $\sigma_0(b) \in X_b$ is contained in exactly two planes in each singular quadric Q_r^b of Q and the intersections I_1^b, I_2^b of these planes with X_b are of dimension 1. Those curves I_1^b, I_2^b can be glued together to form a scheme V of dimension 2. We can define a map $w : V \to \mathrm{Prym}(\tilde{D} \to D)$ by $w(p) = s_1 - s_2$, where $s_1 \in \tilde{D}_{\pi(p_i)}$ is the point corresponding to the family of planes on which p lies and $s_2 \in \tilde{D}_{\pi(p_i)}$ is point corresponding to the other family of planes on the same singular quadric.

One can then look at the intersection product $V \cdot \sigma$ for any section $\sigma \in S_\pi$. If $V \cdot \sigma = p_1 + \cdots + p_n$ for some points $p_1, \ldots, p_n \in V$, we set $u(\sigma) = \sum_i w(p_i)$ and thus obtain a map $u : S_\pi \to \mathrm{Prym}(\tilde{D} \to D)$ to the abelian variety $\mathrm{Prym}(\tilde{D} \to D)$.

As we will see in the example of fibrations over a projective line $\mathbb{P}_k^1$ over a field, studying morphisms from a space of sections $\sigma : \mathbb{P}_{\overline{k}}^1 \to X_{\overline{k}}$ to abelian varieties can help to show the existence of sections over the base field.

Figure 5.1: Sketch of the curves $\tilde{D} \to D \to B$.
The intersection points correspond to singular fibers of π.



5.1 Over $\mathbb{P}^1_k$

We will now devote our attention to del Pezzo fibrations $\pi : X \to \mathbb{P}^1_k$ with square-free discriminant for some field k with $\mathrm{char}(k) \neq 2$. From Theorem 2.4.1, we know that we at least don't have to fear any local obstructions in this case.

This section closely follows [HT14].

Definition 5.1.1. The height of a quadric del Pezzo fibration $\pi : X \to \mathbb{P}^1_k$ with square-free discriminant is $h(\pi) := -2\deg(\pi_*\omega_\pi^{-1})$ where ω_π is the relative dualizing sheaf.

Remark 5.1.2. The number of singular fibers of a quadric del Pezzo fibration π with square-free discriminant over an algebraically closed field is $2h(\pi)$.

We are first going to construct del Pezzo fibrations $\pi : X \to \mathbb{P}^1$ of various heights.

For $n \geq 0$ and $0 \leq k < 5$ and $s \in \{0, 1\}$ let $X \subseteq \mathbb{P}^1_k \times \mathbb{P}^{4+k}_k$ be the intersection of k forms of bidegree $(1, 1)$, a form of bidegree $(n, 2)$ and a form of bidegree $(n + s, 2)$, which are sufficiently general. Let $\pi : X \to \mathbb{P}^1_k$ be the restriction of the canonical projection. In general, this will be a del Pezzo fibration of degree 4 with square-free discriminant. We have

$$\omega_{\mathbb{P}^1_k \times \mathbb{P}^{4+k}_k \to \mathbb{P}^1_k} \cong \mathcal{O}_{\mathbb{P}^1_k \times \mathbb{P}^{4+k}_k}(0, -5-k),$$

so by the adjunction formula

$$\omega_\pi \cong \mathcal{O}_X(2n+s+k, -1).$$

If we chose sufficiently general forms, we obtain

$$\begin{aligned}\pi_*\omega_\pi^{-1} &\cong \mathcal{O}_{\mathbb{P}^1_k}(-2n-s-k) \otimes (\mathcal{O}_{\mathbb{P}^1_k}^{\oplus 5-k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(1)^{\oplus k}) \\ &\cong \mathcal{O}_{\mathbb{P}^1_k}(-2n-s-k)^{\oplus 5-k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(-2n-s-k+1)^{\oplus k},\end{aligned}$$

so the height is

$$\begin{aligned}h(\pi) &= 2((2n+s+k)(5-k) + (2n+s+k-1)k) \\ &= 2(5(2n+s+k) - k) \\ &= 20n + 10s + 8k.\end{aligned}$$

Remark 5.1.3. The factors of the discriminant of the polynomial $\chi(Q)(r:1)$ correspond to the singular fibers of π. Using this fact, the formula for the height can also be deduced from the homogeneity properties of the discriminant $d(p)$ of a polynomial $p(x)$ when viewed as a polynomial in the coefficients of $p(x)$.

This shows that for all even numbers $h \in \mathbb{Z}_0^+ \setminus \{2, 4, 6, 12, 14, 22\}$ there is a del Pezzo fibration of height h with square-free discriminant, at least if the base field is algebraically closed. Examining special cases, it turns out that in fact all even numbers $h \in \mathbb{Z}_0^+ \setminus \{2\}$ occur in this case.

Let from now on $k = \mathbb{F}_q$ be a finite field. We will now try to show in some cases that π has a section defined over k. Let us start with a very simple example:

Example 5.1.4. For $(k, n, s) = (0, 0, 0)$ (i.e., $h(\pi) = 0$), the variety $X \subseteq \mathbb{P}^1_k \times \mathbb{P}^4_k$ is the intersection of two forms of bidegree $(0, 2)$, so the fibers of π are all identical. Since every intersection of two quadrics over the finite field k contains a k-point, there must also exist a (constant) k-section for π.

One way to show that $\pi : X \to \mathbb{P}^1_k$ has a section, is to provide a space S' of $\overline{k}$-sections which is defined over k and forms a principal homogeneous space over an abelian variety G. By Lang's theorem, it must then contain a section $\sigma \in S'$ defined over k. The next two examples will illustrate this.

Example 5.1.5. For $(k, n, s) = (1, 0, 0)$ (i.e., $h(\pi) = 8$), the fibers of π are hyperplane sections of the intersection of two quadrics in $\mathbb{P}^5_k$ where the hyperplane varies linearly with $\mathbb{P}^1_k$. The intersection of these hyperplanes is a subspace isomorphic to $\mathbb{P}^3_k \subseteq \mathbb{P}^5_k$.

Thus, the intersection of all the hyperplane sections is a smooth curve of genus 1 (since the canonial bundle is $\mathcal{O}(-3-1) \otimes \mathcal{O}(2) \otimes \mathcal{O}(2) \cong \mathcal{O}(0)$, so $g = \frac{1}{2} \deg \mathcal{O}(0) + 1 = 1$). Such a curve can be turned into an abelian variety (an elliptic curve) over the algebraic closure. By Theorem 2.3.1, there must then be a k-point on this intersection, hence a (constant) k-section of π.

Example 5.1.6. For $(k, n, s) = (0, 0, 1)$ (i.e., $h(\pi) = 10$), the fibration is a pencil of quadric sections of a quadric that varies linearly with $\mathbb{P}^1_k$. More explicitly, the fibration π can be described as follows: There are three quadratic forms $\mathfrak{q}_\infty, \mathfrak{a}, \mathfrak{b}$ over k in 5 variables such that k-sections of π correspond to homogeneous polynomials $p(t, u)$ satisfying $\mathfrak{q}_\infty(p(t, u)) = u\mathfrak{a}(p(t, u)) + t\mathfrak{b}(p(t, u)) = 0$ for all $t, u \in k$.

The intersection of the quadrics Q_∞, A, B corresponding to the quadric forms $\mathfrak{q}_\infty, \mathfrak{a}, \mathfrak{b}$ is a curve C of genus 5 (since the canonical bundle is $\mathcal{O}(-4-1) \otimes \mathcal{O}(2) \otimes \mathcal{O}(2) \otimes \mathcal{O}(2) \cong \mathcal{O}(1)$, so $g = \frac{1}{2} \deg \mathcal{O}(1) + 1 = \frac{8}{2} + 1 = 5$). It is unclear whether this curve contains a k-point. Assume it doesn't (otherwise, π has a constant section). At least, Theorem 2.3.3 tells us that $\mathrm{Pic}^7(C)$ is non-empty. We will now try to construct a section of π from any such element $d \in \mathrm{Pic}^7(C)$. The strategy is to show that there is a unique (effective) divisor $D \in d$ such that $D = R \cap C$ for some rational degree 4 curve $R \subseteq Q_\infty$ and that this curve R is defined over our base field k. This is motivated by the elementary theorem following this example, which shows that for any "general" divisor $D \in d$, there is exactly one degree 4 curve R with $D = R \cap C$.

It is sufficient to show that there exists a rational degree 4 curve $R \subseteq Q_\infty$ such that $R \cap C \in d$: assume we are given such a curve, parametrized by $p(t, u)$. We are going to show that there is a reparametrization $p(et + fu, gt + hu)$ of $p(t, u)$ that also solves $uA + tB = 0$. The equation we have to solve is

$$u\mathfrak{a}(p(et + fu, gt + hu)) + t\mathfrak{b}(p(et + fu, gt + hu)) = 0$$

But since $\mathfrak{a}(p(et + fu, gt + hu))$ and $\mathfrak{b}(p(et + fu, gt + hu))$ are polynomials of degree 8 sharing a factor of degree 7 (since the intersection of p and A and B has degree 7), we just have to find a non-trivial solution to

$$us_1(et + fu, gt + hu) + ts_2(et + fu, gt + hu) = 0$$

for some (given) homogeneous polynomials s_1 and s_2 of degree 1. This boils down to a system of 3 linear equations (one for the coefficient of each t^2, tu, u^2) and 4 unknowns (the variables e, f, g, h). Hence, since this system has the trivial solution $e = f = g = h = 0$, it must also have a non-trivial solution.

The natural way to study divisors from the divisor class d is to use the map induced by its complete linear system. The Riemann-Roch theorem implies that $l(d) - l(K - d) = \deg(d) + 1 - g = 7 + 1 - 5 = 3$. If $l(K - d) > 0$, then the divisor class $K - d$ (which has degree 1) contains an effective divisor. This would imply that C contains a k-point. Hence, $l(K - d) = 0$, so $l(d) = 3$. The linear system $|d|$ therefore induces a rational map $f : C \dashrightarrow \mathbb{P}^2_k$. In general, its image $C' = f(C)$ is a curve of degree 7 with 10 double points $x_1, \ldots, x_{10}$ in general position. Each divisor $D \in d$ corresponds to a line in $\mathbb{P}^2_k$. The space of polynomials of degree 4 on $\mathbb{P}^2_k$ has dimension $\binom{6}{2} = 15$, so the space of polynomials of degree 4 on $\mathbb{P}^2_k$ with roots at the 10 points $x_1, \dots, x_{10}$ has dimension $15 - 10 = 5$ in general. The space of these polynomials therefore induces a rational map $g : \mathbb{P}^2_k \dashrightarrow \mathbb{P}^4_k$. By choosing an appropriate base, we can ensure that $\overline{g(C')} = C$. The space $Q'_\infty := \overline{g^{-1}(Q_\infty)} \subseteq \mathbb{P}^2$ is given by a (nonzero) homogeneous equation of degree 8 (since g has degree 4 and Q_∞ has degree 2). But since $\overline{g(C')} = C \subseteq Q_\infty$, the curve C' of degree 7 is contained in Q'_∞. Therefore, we must have $Q'_\infty = C' \cup l$ for some line $l \subseteq \mathbb{P}^2_k$. The image $\overline{g(l)} \subseteq \mathbb{P}^4$ of this line is a curve of degree 4 such that $C \cap \overline{g(l)} = f^{-1}(l) \in d$ and $\overline{g(l)} \subseteq Q_\infty$.

Theorem 5.1.7 ([CLSBSD99, Corollary 1.4 and Remark 1.6])**.** *Let k be a field, $n \geq 2$ and $P_0, \dots, P_{n+2} \in \mathbb{P}^n_k$ be $n+3$ points in general linear position (i.e., any $m \leq n+1$ of the points span a linear space of dimension $m-1$). Then, there is a unique rational curve of degree n passing through these $n+3$ points and this curve is smooth.*

Proof of the theorem. Without limit of generality, we can assume that the first $n+1$ points form a standard basis, i.e., we have

$$P_i = (0 : \cdots : 0 : 1 : 0 : \cdots : 0) \qquad \text{for all } 0 \leq i \leq n$$

where the value at the i-th index (counting from 0) is 1.

A parametrization of a rational curve of degree n is given by homogeneous polynomials $p_i(T, U)$ of degree n for all $i \in \{0, \dots, n\}$ that don't all have a non-trivial zero in common (yielding $p(T, U) = (p_0(T, U) : \cdots : p_n(T, U))$). That such a curve passes through $P_0, \dots, P_{n+2}$ means that there are distinct $(t_1 : u_1), \dots, (t_{n+3} : u_{n+3}) \in \mathbb{P}^1_k$ such that

$$p(t_j, u_j) = (p_0(t_j, u_j) : \cdots : p_n(t_j, u_j)) = P_j \qquad \text{for all } 0 \leq j \leq n+2.$$

For the points P_j with $0 \leq j \leq n$, this is equivalent to $p_i(t_j, u_j) = 0$ for all $0 \leq i, j \leq n$ with $i \neq j$. But that means that we can split off n linear factors $u_j T - t_j U$ from each polynomial $p_i(T, U)$. Since $p_i(T, U)$ only has degree n though, the remaining factor is a constant. Hence, there should exist $\lambda_i \in k^\times$ such that

$$p_i(T, U) = \lambda_i \prod_{0 \leq j \leq n,\ i \neq j} (u_j T - t_j U) \qquad \text{for each } 0 \leq i \leq n.$$

Dividing t_i and u_i by λ_i, we can in fact even assume that $\lambda_i = 1$ for all $0 \leq i \leq n$.

Let us now look at the two remaining conditions: $p(t_j, u_j) = P_j$ for $j \in \{n+1, n+2\}$. We are now going to apply a coordinate transformation to (a dense subset of) $\mathbb{P}^n$ to simplify these conditions. Let $P_j = (x_{0,j} : \cdots : x_{n,j})$. The two conditions obviously imply

$$\left[\prod_{i \neq i_0} x_{i,j}\right]_{i_0} = \left[\prod_{i \neq i_0} p_i(t_j, u_j)\right]_{i_0} = [u_{i_0} t_j - t_{i_0} u_j]_{i_0} \in \mathbb{P}^n_k$$

for $j \in \{n+1, n+2\}$. In fact, they are also sufficient since the points $P_0, \dots, P_{n+2}$ are in general linear position, and thus the coordinates of P_{n+1} and P_{n+2} are all non-zero.

These two equations simply mean that the line in $\mathbb{P}^n$ defined by

$$(T, U) \mapsto [u_{i_0} T - t_{i_0} U]_{0 \le i_0 \le n}$$

goes through the point $Q_j := \left[\prod_{i \neq i_0} x_{i,j}\right]_{i_0}$ at (t_j, u_j) for $j \in \{n+1, n+2\}$. But, up to reparametrization, there is obviously exactly one line through the (distinct) points Q_{n+1} and Q_{n+2}. □

Remark 5.1.8. A special case of this theorem is the famous fact that five points in general linear position in the plane determine a conic.

Sometimes, it is hard or impossible to find a space S' of $\overline{k}$-sections which is defined over k and where each point on an abelian variety G corresponds to exactly one section. If there are multiple, but finitely many sections $\sigma \in S'$ for each $g \in G_{\overline{k}}$, we are in trouble: those might exactly be Galois conjugates and thus not be defined over the ground field k. But sometimes, one can use the following theorem due to Esnault.

Theorem 5.1.9 ([Esn03, Corollary 1.3])**.** *Let Y be a smooth, projective and geometrically connected variety over the finite field $k = \mathbb{F}_q$ and assume that Y is chain rationally connected over the algebraic closure $\overline{k(Y)}$. Then, the variety Y contains a k-point.*

This shows that it is sufficient to find a space S' of $\overline{k}$-sections which is defined over k and a morphism $S' \to P$ where P is some principal homogeneous space of an algebraic variety G such that every fiber of $S' \to P$ is smooth, projective, geometrially connected and chain rationally connected over the algebraic closure.

5.2 Over $\mathbb{Z}$

Let us look at fibrations with base $B = \operatorname{Spec} \mathbb{Z}$, now. Let

$$X = \operatorname{Proj} \mathbb{Z}[x_1, \ldots, x_5] / (\mathfrak{q}_\infty(x_1, \ldots, x_5), \mathfrak{q}_0(x_1, \ldots, x_5))$$

for two integral quadratic forms $\mathfrak{q}_\infty$ and $\mathfrak{q}_0$ corresponding to the matrices Q_∞ and Q_0.
Let us for example assume that $\chi(Q)(u : 1)$ is monic of degree 5 and irreducible and that $\varepsilon(Q_r) \neq 1 \in \mathbb{Q}(r)^\times / \mathbb{Q}(r)^{\times 2}$ for any root r of $\chi(Q)(u : 1)$. Let $K = \mathbb{Q}(r)$ and $L = K(\sqrt{\varepsilon(Q_r)})$ for some such root r. For simplicity, also assume that the rings $\mathbb{Z}[r]$ and $\mathbb{Z}[r, \sqrt{\varepsilon(Q_r)}]$ are integrally closed.
Then, the curves $\tilde{C}$ and C are the spectra of the set of integers in L and K, respectively:

$$\tilde{C} = \operatorname{Spec} \mathcal{O}_L, \qquad C = \operatorname{Spec} \mathcal{O}_K.$$

Our maps $\tilde{C} \to C \to B$ are simply the ones induced by the inclusions $\mathbb{Z} \hookrightarrow \mathcal{O}_K \hookrightarrow \mathcal{O}_L$.
One can use Arakelov theory to obtain the groups $\operatorname{Pic}^0_{\mathrm{Ar}}(\tilde{C})$ and $\operatorname{Pic}^0_{\mathrm{Ar}}(C)$ and a replacement for the Prym variety $\operatorname{Prym}(\tilde{D} \to D)$.

You can again use Theorem 1.5.1 to construct examples of del Pezzo surfaces with given characteristic form $\chi(Q)$ if $\chi(Q)(r:1)$ is a monic polynomial of degree 5 with constant term ± 1.

We've done some computer experiments regarding the minimal height of solutions. For example, all del Pezzo surfaces with

$$Q_\infty = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix}$$

and a matrix Q_0 with integer coefficients between -1 and 1 such that the characteristic form is irreducible and has only real roots (this means that the Galois group is as large as possible and the Brauer group is trivial) have either a local obstruction or a $\mathbb{Q}$-point of height at most 2.

Bibliography

[Ben68] Edward Bender. Characteristic polynomials of symmetric matrices. *Pacific Journal of Mathematics*, 25(3):433–441, 1968.

[Bor08] Iurie Boreico. My Favorite Problem: Linear Independence of Radicals. *The Harvard College Mathematics Review*, 2(1):87–92, 2008.

[CLSBSD99] Daniel F. Coray, Donald J. Lewis, Nicholas I. Shepherd-Barron, and Peter Swinnerton-Dyer. Cubic threefolds with six double points. *Number theory in progress*, 1:63–74, 1999.

[dJ04] Robin S. de Jong. *Explicit Arakelov geometry*. PhD thesis, University of Amsterdam, 2004.

[Esn03] Hélene Esnault. Varieties over a finite field with trivial Chow group of 0-cycles have a rational point. *Inventiones mathematicae*, 151(1):187–191, 2003.

[GS92] Henri Gillet and Christophe Soulé. An arithmetic Riemann-Roch theorem. *Inventiones mathematicae*, 110(1):473–543, 1992.

[Har77] Robin Hartshorne. *Algebraic geometry*, volume 52. Springer, 1977.

[HS00] Marc Hindry and Joseph H. Silverman. *Diophantine Geometry: An Introduction*. Graduate Texts in Mathematics. Springer, 2000.

[HT14] Brendan Hassett and Yuri Tschinkel. Quartic Del Pezzo surfaces over function fields of curves. *Central European Journal of Mathematics*, 12(3):395–420, 2014.

[Lan55] Serge Lang. Abelian varieties over finite fields. *Proceedings of the National Academy of Sciences of the United States of America*, 41(3):174, 1955.

[Lan88] Serge Lang. *Introduction to Arakelov theory*. Springer, 1988.

[Lor08] Falko Lorenz. *Algebra: Volume II: Fields with Structure, Algebras and Advanced Topics*, volume 2. Springer, 2008.

[Mil08] James S. Milne. Abelian Varieties (v2.00), 2008. Available at `www.jmilne.org/math/`.

[Mil12] James S. Milne. Algebraic Geometry (v5.22), 2012. Available at `www.jmilne.org/math/`.

[Neu92] Jürgen Neukirch. *Algebraische Zahlentheorie*, volume 1. Springer Berlin etc., 1992.

[Rei72] Miles A. Reid. *The complete intersection of two or more quadrics*. PhD thesis, University of Cambridge, 1972.

[Ros02] Michael Rosen. *Number theory in function fields*, volume 210. Springer, 2002.

[Sch31] Friedrich Karl Schmidt. Analytische Zahlentheorie in Körpern der Charakteristik p. *Mathematische Zeitschrift*, 33(1):1–32, 1931.

[Sch13] Christoph Schießl. Computing the Brauer Groups of Quartic del Pezzo Surfaces. Master's thesis, Ludwig-Maximilians-Universität München, 2013. Available at `http://www.iazd.uni-hannover.de/~derenthal/papers/schiessl-master.pdf`.

[VAV12] Anthony Várilly-Alvarado and Bianca Viray. Vertical Brauer groups and del Pezzo surfaces of degree 4. *arXiv preprint arXiv:1210.0920*, 2012.

[Wit07] Olivier Wittenberg. *Intersections de deux quadriques et pinceaux de courbes de genre 1*. Springer Heidelberg, 2007.

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